"MODERN" INTRODUCTORY ANALYSIS

(DOLCIANI)

SCRATCH PAD 3

SOLUTIONS TO EXERCISES

4. The Algebra of Vectors 4.6 – 4.8

5. Plane Analytic Geometry of Points and Lines 5.1 – 5.9

MODERN INTRODUCTORY ANALYSIS

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c. 1964, 1967, 1970, 1974, 1977, 1980, 1986 . . .

4. The Algebra of Vectors
   Parallel and Perpendicular Vectors

$\boxed{4\text{-}5}$ Exercises 59–62

59 Prove: $\vec{v}$ and $\vec{t}$ have the same direction if and only if $\|\vec{v}+\vec{t}\| = \|\vec{v}\| + \|\vec{t}\|$

I want to proceed in an unrestricted manner and only after scratch work, organizing into some kind of proof.

$$\|\vec{v}+\vec{t}\| = \sqrt{(v_1+t_1)^2 + (v_2+t_2)^2}$$
$$= \sqrt{v_1^2 + 2t_1v_1 + t_1^2 + v_2^2 + 2v_2t_2 + t_2^2}$$

$$\|\vec{v}\| + \|\vec{t}\| = \sqrt{v_1^2+v_2^2} + \sqrt{t_1^2+t_2^2}$$

If $\|\vec{v}+\vec{t}\| = \|\vec{v}\| + \|\vec{t}\|$, when squaring both sides

$$\|\vec{v}+\vec{t}\|^2 = (\|\vec{v}\| + \|\vec{t}\|)^2$$

$$v_1^2+v_2^2+t_1^2+t_2^2+2v_1t_1+2v_2t_2 =$$

$$2\sqrt{(v_1^2+v_2^2)(t_1^2+t_2^2)} + t_1^2+t_2^2+v_1^2+v_2^2$$

$$2v_1t_1 + 2v_2t_2 = 2\sqrt{(v_1^2+v_2^2)(t_1^2+t_2^2)}$$
$$= 2\cdot\sqrt{v_1^2t_1^2 + v_1^2t_2^2 + v_2^2t_1^2 + v_2^2t_2^2}$$

$$v_1t_1 + v_2t_2 = \sqrt{v_1^2t_1^2 + v_1^2t_2^2 + v_2^2t_1^2 + v_2^2t_2^2}$$

Again, we square both sides:

$$v_1^2t_1^2 + 2v_1v_2t_1t_2 + v_2^2t_2^2 = v_1^2t_1^2 + v_1^2t_2^2 + v_2^2t_1^2 + v_2^2t_2^2$$

$$2v_1v_2t_1t_2 = v_1^2t_2^2 + v_2^2t_1^2$$

$$0 = v_1^2t_2^2 - 2v_1v_2t_1t_2 + v_2^2t_1^2$$

$$0 = (v_1t_2 - v_2t_1)^2$$

$$0 = v_1t_2 - v_2t_1 \longleftrightarrow v_2t_1 = v_1t_2$$

$$v_1 = \left(\frac{v_2}{t_2}\right)t_1$$

$$\vec{v} = \left(\left(\frac{v_2}{t_2}\right)t_1, \left(\frac{v_2}{t_2}\right)t_2\right)$$

$$\vec{v} = \left(\frac{v_2}{t_2}\right)\vec{t}$$

$\frac{v_2}{t_2} > 0$ if $v_2 > 0$ and $t_2 > 0$

$\therefore \frac{v_2}{t_2} > 0$ and $v_1t_2 > 0$

Since $\vec{v}$ is a positive scalar multiple of $\vec{t}$, they have the same direction.

Conversely, if $\vec{v}$ and $\vec{t}$ have the same direction $\vec{v} = k\vec{t}$ where $k > 0$.

$\|\vec{v} + \vec{t}\| = \|k\vec{t} + \vec{t}\| = \|(k+1)\vec{t}\|$

$= (k+1)\|\vec{t}\| = k \cdot \|\vec{t}\| + \|\vec{t}\|$

$= \|\vec{v}\| + \|\vec{t}\|$

(60) Prove: If $\vec{v}$ and $\vec{t}$ have opposite directions, then $\|\vec{v} + \vec{t}\| < \|\vec{v}\| + \|\vec{t}\|$

If $\vec{v} = k\vec{t}$, $k < 0$ and $-k > 0$,

$\|\vec{v} + \vec{t}\| = \|k\vec{t} + \vec{t}\| = |k+1| \cdot \|\vec{t}\|$

$\|\vec{v}\| + \|\vec{t}\| = \|k \cdot \vec{t}\| + \|\vec{t}\| = |k| \cdot \|\vec{t}\| + \|\vec{t}\|$

$= (-k+1) \cdot \|\vec{t}\|$

$k < 0 < -k \longleftrightarrow k+1 < 1-k$, and since $k < 0$

$|k+1| < -k+1$ and $\|\vec{v} + \vec{t}\| < \|\vec{v}\| + \|\vec{t}\|$

(61) Given: $L = \{(x,y) : (x,y) = (0,1) + k(1,1), k \in R\}$

Show that $(x,y) \in L$ if and only if $y - x = 1$.

If $x, y \in R$ and $y - x = 1$, $y = x + 1$

$$(x,y) = (x, x+1) = (x,x) + (0,1)$$
$$= (0,1) + x(1,1) \qquad [\text{let } x = k]$$

If $(x,y) \in L$ then $(x,y) = (0,1) + k(1,1)$
$= (0+k, 1+k)$,

$x = 0 + k$, $y = 1 + k$
$y = x + 1$, $y - x = 1$

(62) Prove: If $\vec{v}$ and $\vec{t}$ are a pair of nonzero, nonparallel vectors, and a and b are scalars such that $a\vec{v} + b\vec{t} = \vec{0}$, then $a = b = 0$.

$a\vec{v} + b\vec{t} = 0 \quad \therefore \quad a \cdot \vec{v} = -b \cdot \vec{t}$

if $a \neq 0$, then $\vec{v} = -\frac{b}{a}\vec{t}$

but $\vec{v}$ and $\vec{t}$ are not parallel, that means that if $b \neq 0$, $\vec{t} = -\frac{a}{b}\vec{v}$

It must be, therefore, that $a = b = 0$.

4-6 Inner Product

Given $\vec{v}$ and $\vec{u}$ are vectors and r and s are scalars.
State which of the following are vectors and which are scalars.

(1) $-\vec{v}$ is a vector
(2) $\|\vec{u}\|$ is a scalar
(3) $\vec{u}+\vec{v}$ is a vector
(4) $\vec{u}\cdot\vec{v}$ is a scalar
(5) $r\cdot\vec{u}$ is a vector
(6) $r\cdot s$ is a scalar
(7) $\|\vec{u}+\vec{v}\|$ is a scalar
(8) $s(\vec{u}+\vec{v})$ is a vector
(9) $\vec{v}(r+s)$ is a vector
(10) $\|\vec{v}\|+\|\vec{u}\|$ is a scalar
(11) $\vec{v}\cdot(\vec{v}+\vec{u})$ is a scalar.
(12) $|r+s|$ is a scalar

Find each inner product

(13) $(3,2)\cdot(5,-1) = 3\cdot 5 + 2(-1) = 15-2 = 13$
(14) $(2,-1)\cdot(3,4) = 2\cdot 3 + (-1)\cdot 4 = 6-4 = 2$
(15) $(0,3)\cdot(3,0) = 0\cdot 3 + 3\cdot 0 = 0$
(16) $(\sqrt{2},5)\cdot(\sqrt{2},-3) = \sqrt{2}\cdot\sqrt{2} + 5\cdot(-3) = 2-15 = -13$
(17) $(2\sqrt{3},\sqrt{3})\cdot(\sqrt{2},4\sqrt{3}) = 2\sqrt{3}\cdot\sqrt{2} + \sqrt{3}\cdot 4\sqrt{3}$
$= 2\sqrt{6} + 12$
(18) $(a,b)\cdot(c,d) = a\cdot c + b\cdot d$

State whether the given vectors are perpendicular or nonperpendicular

(19) $(3,-1)$; $(2,-6)$. Since $3\cdot 2 + (-1)(-6) = 6+6 = 12 \neq 0$, these are nonperpendicular
(20) $(-2,8)$; $(-3,-\frac{3}{4})$. Since $(-2)\cdot(-3) + 8(-\frac{3}{4})$
$= 6-6 = 0$, these are perpendicular.

(21) $\left(\frac{2}{3}, -\frac{1}{4}\right); \left(-\frac{3}{2}, 4\right)$

Since $\left(\frac{2}{3}\right)\left(-\frac{3}{2}\right) + \left(-\frac{1}{4}\right) 4 = -1 - 1 = -2 \neq 0$, these are nonperpendicular

(22) $(6, 2\sqrt{3}); (\sqrt{3}, -3)$

Since $6\sqrt{3} + 2\sqrt{3} \cdot (-3) = 6\sqrt{3} - 6\sqrt{3} = 0$, these are perpendicular

(23) $\vec{r}; -\vec{r} \quad (\vec{r} \neq \vec{0})$

Since $r_1 \cdot (-r_1) + r_2(-r_2) = -r_1^2 - r_2^2 \neq 0$, these are nonperpendicular

(24) $(a, 0); (0, -a) \quad (a \neq 0)$

Since $a \cdot 0 - 0\,a = 0$, these are perpendicular

Find k if the given vectors are (a) perpendicular (b) parallel

(25) $(3, 4); (6, k)$

(a) $3 \cdot 6 + 4k = 0 \iff 4k = -18 \iff k = -\frac{9}{2}$

(b) Since $6 = 2 \cdot 3$, $k = 2 \cdot 4 = 8$

(26) $(-2, 5); (10, k)$

(a) $(-2)10 + 5k = 0 \iff 5k = 20 \iff k = 4$

(b) Since $10 = -5(-2)$, $k = -5(5) = -25$

(27) $(k, 2)$; $(\sqrt{3}, 6)$

(a) $k \cdot \sqrt{3} + 2 \cdot 6 = 0 \leftrightarrow k\sqrt{3} = -12 \leftrightarrow k = -\frac{12}{\sqrt{3}}$

$= -\frac{12\sqrt{3}}{3} = -4\sqrt{3}$

(b) Since $2 = \frac{1}{3}(6)$, $k = \frac{1}{3}(\sqrt{3}) = \frac{\sqrt{3}}{3}$

(28) $(2, 4)$; $(k, \sqrt{3})$

(a) Since $\sqrt{3} = \frac{\sqrt{3}}{4}(4)$, $k = \frac{\sqrt{3}}{4}(2) = \frac{\sqrt{3}}{2}$

(29) $(\sqrt{3}, k)$; $(2, 6)$

(a) $2\sqrt{3} + 6k = 0 \leftrightarrow 6k = -2\sqrt{3} \leftrightarrow k = -\frac{\sqrt{3}}{3}$

(b) Since $\sqrt{3} = \frac{\sqrt{3}}{2}(2)$, $k = \frac{\sqrt{3}}{2}(6) = 3\sqrt{3}$

(30) $(4, 3)$; $(-3, k)$

(a) $4(-3) + 3k = 0 \leftrightarrow 3k = 12 \leftrightarrow k = 4$

(b) Since $-3 = -\frac{3}{4}(4)$, $k = \left(-\frac{3}{4}\right)(3) = -\frac{9}{4}$

Note: Do part (b) over a different way and compare results

(25) $(3, 4)$; $(6, k)$: $(6, k) = r(3, 4)$

$6 = 3r$, $k = 4r$, $r = 2$, $k = 4 \cdot 2 = 8$

same as first method

(26) $(-2, 5)$; $(10, k)$: $(10, k) = r(-2, 5)$

$10 = -2r$, $k = 5r$, $r = -5$, $k = 5(-5) = -25$

Same as first method.

(27) $(k,2);(\sqrt{3},6)$: $(k,2) = r(\sqrt{3},6)$

$2r = 6$, $k = \sqrt{3}\cdot r$, $r = \frac{1}{3}$, $k = \frac{\sqrt{3}}{3}$

same

(28) $(2,4);(k,\sqrt{3})$: $(k,\sqrt{3}) = r(2,4)$

$k = 2r$, $\sqrt{3} = 4r$, $r = \frac{\sqrt{3}}{4}$, $k = 2\left(\frac{\sqrt{3}}{4}\right) = \frac{\sqrt{3}}{2}$

same as first method

(29) $(\sqrt{3},k);(2,6)$: $(\sqrt{3},k) = r(2,6)$

$\sqrt{3} = 2r$, $k = 6r$, $r = \frac{\sqrt{3}}{2}$, $k = 6\left(\frac{\sqrt{3}}{2}\right) = 3\sqrt{3}$

same as first method

(30) $(4,3);(-3,k)$: $(-3,k) = r(4,3)$

$-3 = 4r$, $k = 3r$, $r = \frac{-3}{4}$, $k = 3\left(\frac{-3}{4}\right) = \frac{-9}{4}$

same as first method

Show that the specified subsets of $\mathbb{R}\times\mathbb{R}$ are equal.

(31) $\{(x,y) : (x,y) \text{ and } (4,-5) \text{ are perpendicular}\}$;
$\{(x,y) : y = \frac{4}{5}x\}$

$4x - 5y = 0 \longleftrightarrow 4x = 5y \longleftrightarrow y = \frac{4}{5}x$

In words: (x,y) and $(4,-5)$ are perpendicular if and only if $0 = (x,y)\cdot(4,-5) = 4x-5y$.

This is equivalent to $y = \frac{4}{5}x$, which is the condition of the second set.

(32) $\{(x,y) : (x,y) \text{ and } (5,-2) \text{ are parallel}\}$;
$\{(x,y) : x = 5k,\ y = -2k;\ k \in R\}$

(x,y) and $(5,-2)$ are parallel if $(x,y) = k(5,-2)$
$\leftrightarrow$ $x = 5k$ and $y = -2k$, which is the condition of the second set.

Let $\vec{r} = (-2,3)$, $\vec{s} = (3,-2)$, and $\vec{t} = (4,-1)$.
Verify each statement.

(33) $\vec{r}\cdot\vec{s} = \vec{s}\cdot\vec{r}$; $\vec{r}\cdot\vec{s} = (-2,3)\cdot(3,-2) = (-2)\cdot 3 + 3\cdot(-2) = -12$
$\vec{s}\cdot\vec{r} = (3,-2)\cdot(-2,3) = 3(-2)+(-2)3 = -6-6 = -12$

(34) $3(\vec{s}\cdot\vec{t}) = (3\vec{s})\cdot\vec{t}$
$3(\vec{s}\cdot\vec{t}) = 3\left[(3,-2)\cdot(4,-1)\right] = 3(12+2) = 42$
$(3\vec{s})\cdot\vec{t} = 3(3,-2)\cdot(4,-1) = (9,-6)\cdot(4,-1)$
$= 9\cdot 4 + (-6)(-1) = 36+6 = 42$

(35) $\vec{t}\cdot(\vec{r}+\vec{s}) = \vec{t}\cdot\vec{r} + \vec{t}\cdot\vec{s}$
$\vec{t}\cdot(\vec{r}+\vec{s}) = (4,-1)\cdot\left[(-2,3)+(3,-2)\right] = (4,-1)\cdot(1,1)$
$= 4-1 = 3$
$\vec{t}\cdot\vec{r} + \vec{t}\cdot\vec{s} = (4,-1)\cdot(-2,3) + (4,-1)\cdot(3,-2) = -11+14 = 3$

(36) $\vec{s}\cdot(\vec{t}-\vec{r}) = \vec{s}\cdot\vec{t} - \vec{s}\cdot\vec{r}$ (40)

$\vec{s}\cdot(\vec{t}-\vec{r}) = (3,-2)\cdot[(4,-1)-(-2,3)]$

$= (3,-2)\cdot[(4,-1)+(2,-3)] = (3,-2)\cdot(6,-4)$

$= 3\cdot 6 + (-2)(-4) = 18+8 = 26$ (41)

$\vec{s}\cdot\vec{t} - \vec{s}\cdot\vec{r} = (3,-2)\cdot(4,-1) - (3,-2)\cdot(-2,3)$

$= [3\cdot 4 + (-2)(-1)] - [3(-2)+(-2)(3)]$

$= (12+2) - (-6-6) = 14+12 = 26$ (42)

In each exercise find a vector $\vec{v} \neq \vec{0}$ such that $\vec{r}\cdot\vec{v} = 0$

(37) $\vec{r} = (0,2)$: $(0)\cdot v_1 + 2(v_2) = 0$

$v_2 = 0$ and $v_1 \in \mathbb{R}$, $v_1 \neq 0$

(38) $\vec{r} = (-1,3)$: $(-1)\cdot v_1 + 3v_2 = 0$ (43)

$3v_2 = v_1$; $\vec{v} = (3v_2, v_2)$; $v_2 \in \mathbb{R}$, $v_2 \neq 0$

(39) $\vec{r} = (\sqrt{2},2)$: $\sqrt{2}\cdot v_1 + 2v_2 = 0$

$\sqrt{2}\,v_1 = -2v_2$; $v_2 = -\frac{\sqrt{2}}{2}v_1$ (44)

$\vec{v} = \left(v_1, -\frac{\sqrt{2}}{2}v_1\right)$ (45)

or $v_1 = \frac{-2}{\sqrt{2}}v_2 = \frac{-2\sqrt{2}}{2}v_2 = -\sqrt{2}\,v_2$

$\vec{v} = (-\sqrt{2}\,v_2, v_2)$, $v_2 \neq 0$, $v_2 \in \mathbb{R}$

(40) $\vec{r} = (0, \sqrt{3})$: $0 \cdot v_1 + \sqrt{3} v_2 = 0$; $\sqrt{3} v_2 = 0$

so $v_2 = 0$. $\vec{v} = (v_1, 0)$, $v_1 \in R$, $v_1 \neq 0$

Given : $\vec{v}, \vec{t}$ elements of $R \times R$, $r \in R$.
Prove the given property of inner products.

(41) Commutative : $\vec{v} \cdot \vec{t} = \vec{t} \cdot \vec{v}$

$\vec{v} = (v_1, v_2)$; $\vec{t} = (t_1, t_2)$

$\vec{v} \cdot \vec{t} = v_1 t_1 + v_2 t_2 = t_1 v_1 + t_2 v_2 = \vec{t} \cdot \vec{v}$

(42) Associative : $r(\vec{v} \cdot \vec{t}) = (r\vec{v}) \cdot \vec{t}$

$r(\vec{v} \cdot \vec{t}) = r[(v_1, v_2) \cdot (t_1, t_2)]$

$= r(v_1 t_1 + v_2 t_2) = r \cdot v_1 t_1 + r v_2 t_2$

$= (r v_1) t_1 + (r \cdot v_2) t_2 = r(v_1, v_2) \cdot (t_1, t_2)$

$= (r\vec{v}) \cdot \vec{t}$

(43) Norm : $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$

$(v_1, v_2) \cdot (v_1, v_2) = v_1^2 + v_2^2 = \left(\sqrt{v_1^2 + v_2^2}\right)^2$

$= \|\vec{v}\|^2$

(44) $\vec{v} \cdot \vec{0} = 0$: $(v_1, v_2) \cdot (0, 0) = v_1(0) + v_2(0) = 0$

(45) Show that $[(2,3) \cdot (5,1)] \cdot (-2,4)$
$\neq (2,3)[(5,1) \cdot (-2,4)]$

$$[(2,3)\cdot(5,1)](-2,4) = (2\cdot 5+3\cdot 1)(-2,4)$$

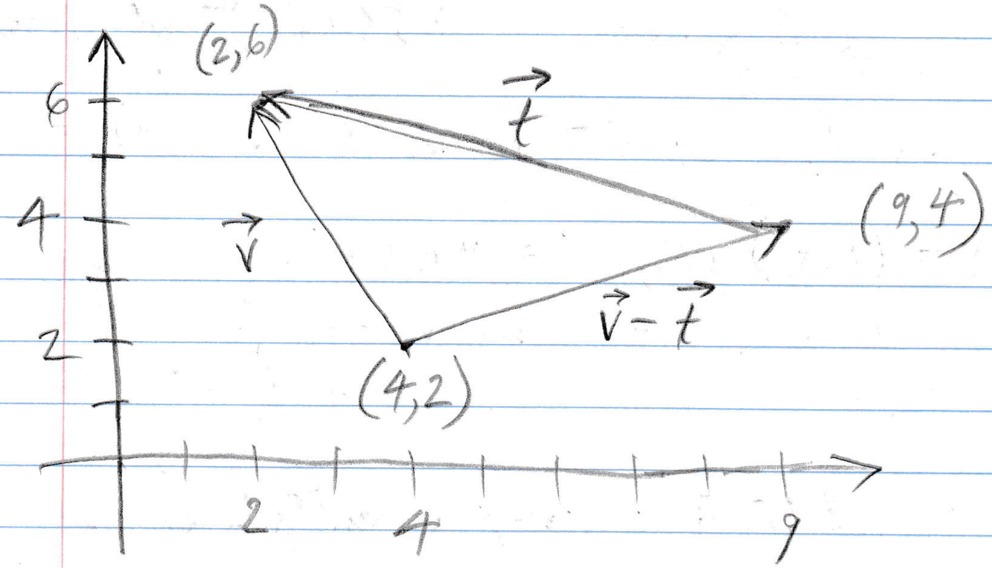
$$= 13(-2,4) = (-26, 52)$$

$$\neq (2,3)[(5,1)\cdot(-2,4)] = (2,3)(5(-2)+1\cdot 4)$$

$$= (2,3)(-10+4) = (2,3)(-6) = (-12,-18)$$

(46) In the given vector representation the ordered pairs are coordinates of points. Find $\vec{v}$ and $\vec{t}$ and show that:

$$2\vec{v}\cdot\vec{t} = \|\vec{v}\|^2 + \|\vec{t}\|^2 - \|\vec{v}-\vec{t}\|^2$$



$\vec{v} = (2,6)-(4,2) = (-2,4)$

$\vec{t} = (2,6)-(9,4) = (-7,2)$

$\vec{v}-\vec{t} = (9,4)-(4,2) = (5,2) = (-2,4)-(-7,2) = (5,2)$

$2\vec{v}\cdot\vec{t} = 2(-2,4)\cdot(-7,2) = (-4,8)\cdot(-7,2)$

$= (-4)(-7) + 8\cdot 2 = 28 + 16 = 44$

$\|\vec{v}\|^2 = \left(\sqrt{(-2)^2+4^2}\right)^2 = \left(\sqrt{4+16}\right)^2 = \left(\sqrt{20}\right)^2 = 20$

$\|\vec{t}\|^2 = \left(\sqrt{(-7)^2+2^2}\right)^2 = \left(\sqrt{49+4}\right)^2 = 53$

$\|\vec{v}-\vec{t}\|^2 = \|(5,2)\|^2 = \left(\sqrt{5^2+2^2}\right)^2 = 29$

$20+53-29 = 44$

In exercises 47 and 48 prove the given property of inner products assuming $\vec{v}$, $\vec{u}$, $\vec{t}$, and $\vec{s}$ are elements of $R \times R$.

(47) Substitution: If $\vec{v} = \vec{u}$ and $\vec{t} = \vec{s}$, $\vec{v}\cdot\vec{t} = \vec{u}\cdot\vec{s}$

$\vec{v}\cdot\vec{t} = (v_1, v_2)\cdot(t_1, t_2) = v_1 t_1 + v_2 t_2$;

Since $\vec{v} = \vec{u}$, $v_1 = u_1$ and $v_2 = u_2$;

also since $\vec{t} = \vec{s}$, $t_1 = s_1$ and $t_2 = s_2$;

$\therefore$ by substitution property of multiplication in R

$v_1 t_1 + v_2 t_2 = u_1 s_1 + u_2 s_2 = \vec{u}\cdot\vec{s}$

(48) Distributive: $\vec{v}\cdot(\vec{t}+\vec{s}) = \vec{v}\cdot\vec{t} + \vec{v}\cdot\vec{s}$
and $(\vec{t}+\vec{s})\cdot\vec{v} = \vec{t}\cdot\vec{v} + \vec{s}\cdot\vec{v}$

$$(v_1, v_2)\cdot[(t_1, t_2)+(s_1, s_2)] = (v_1, v_2)\cdot(t_1+s_1, t_2+s_2)$$

$$= v_1(t_1+s_1) + v_2(t_2+s_2) = v_1t_1 + v_1s_1 + v_2t_2 + v_2s_2$$

and $(\vec{t}+\vec{s})\cdot\vec{v} = [(t_1, t_2)+(s_1, s_2)]\cdot(v_1, v_2)$

$$= (t_1+s_1, t_2+s_2)\cdot(v_1, v_2)$$

$$= (t_1+s_1)(v_1) + (t_2+s_2)(v_2) = v_1t_1+v_1s_1+v_2t_2+v_2s_2$$

Another way to proceed with such an exercise is to "work in and then back out": *

$$\vec{v}\cdot(\vec{t}+\vec{s}) = (v_1, v_2)\cdot[(t_1, t_2)+(s_1, s_2)]$$

$$= (v_1, v_2)\cdot(t_1+s_1, t_2+s_2) = v_1(t_1+s_1)+v_2(t_2+s_2)$$

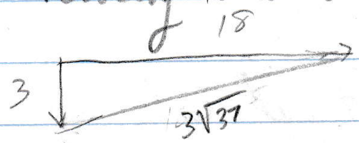
$$= v_1t_1 + v_1s_1 + v_2t_2 + v_2s_2 \overset{*}{=} (v_1t_1 + v_1s_1)+(v_2t_2+v_2s_2)$$

$$= \vec{v}\cdot\vec{t} + \vec{v}\cdot\vec{s}$$

$$(\vec{t}+\vec{s})\cdot\vec{v} = \vec{v}\cdot(\vec{t}+\vec{s}) = \vec{v}\cdot\vec{t} + \vec{v}\cdot\vec{s}$$

$$= \vec{t}\cdot\vec{v} + \vec{s}\cdot\vec{v}$$

(49) A ship sails east at 18 mph. A man walks across the deck directly south at 3 mph. What is his velocity relative to the water?



let $\|\vec{v}\| = 18$

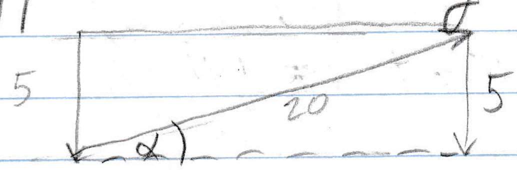
let $\|\vec{t}\| = 3$

$\|\vec{v} + \vec{t}\| = 3\sqrt{37}$

The velocity of the ship is represented by the vector $(18, 0)$; the velocity of the man is represented by the vector $(0, -3)$. The magnitude of the man's velocity relative to the water is $\|(18,0) + (0,-3)\|$

$$= \|(18,-3)\| = \sqrt{18^2 + (-3)^2} = \sqrt{333} = 3\sqrt{37}$$

(50) A man wishes to go by motor boat directly across a river of width 2 miles. The water is moving south at 5 mph and the boat can travel 20 mph. On the basis of a vector diagram, approximate the angle at which he should head the boat upstream if he wishes to reach a point directly opposite his starting point.



$$\alpha = \arcsin\left(\frac{5}{20}\right)$$

$$= \arcsin\left(\frac{1}{4}\right) \approx 0.247404$$

$$\approx 14^\circ$$

Given: $\vec{v}$ and $\vec{t}$ elements of $\mathbb{R} \times \mathbb{R}$.
Prove each assertion.

(51) $\vec{v}$ is perpendicular to $\vec{t}$ if and only if $\|\vec{v}+\vec{t}\| = \|\vec{v}-\vec{t}\|$.

If $\vec{v}$ is perpendicular to $\vec{t}$, then $\vec{v}\cdot\vec{t} = 0$,
that is, $(v_1, v_2)\cdot(t_1, t_2) = 0$

$$v_1 t_1 + v_2 t_2 = 0$$

$$\begin{aligned}
\|v+t\|^2 &= \|\vec{v}\|^2 + \|\vec{t}\|^2 + 2\vec{v}\cdot\vec{t} \\
&= \|\vec{v}\|^2 + \|\vec{t}\|^2 + 2\cdot 0 \\
&= \|\vec{v}\|^2 + \|\vec{t}\|^2 - 2\vec{v}\cdot t = \|\vec{v}-\vec{t}\|^2
\end{aligned}$$

(52) $(\vec{v}-\vec{t})\cdot(\vec{v}+\vec{t}) = \|\vec{v}\|^2 - \|\vec{t}\|^2$

$$\begin{aligned}
&\left[(v_1, v_2) - (t_1, t_2)\right]\cdot\left[(v_1, v_2) + (t_1, t_2)\right] \\
&= (v_1 - t_1, v_2 - t_2)\cdot(v_1 + t_1, v_2 + t_2) \\
&= (v_1 - t_1)(v_1 + t_1) + (v_2 - t_2)(v_2 + t_2) \\
&= (v_1^2 - t_1^2) + (v_2^2 - t_2^2) = (v_1^2 + v_2^2) - (t_1^2 + t_2^2) \\
&= \|\vec{v}\|^2 - \|\vec{t}\|^2
\end{aligned}$$

Another way to proceed with exercise 52:

$$(\vec{v} - \vec{t}) \cdot (\vec{v} + \vec{t}) = (\vec{v} - \vec{t}) \cdot \vec{v} + (\vec{v} - \vec{t}) \cdot \vec{t}$$
$$= (\vec{v} \cdot \vec{v}) + (-\vec{t} \cdot \vec{v}) + (\vec{v} \cdot \vec{t}) + (-\vec{t} \cdot \vec{t})$$

applying the distributive law of multiplication over addition for vectors to the vector $-\vec{t}$

Since $-\vec{t} \cdot \vec{v} = (-1)\,\vec{v} \cdot \vec{t}$, $-\vec{t} \cdot \vec{v} + \vec{v} \cdot \vec{t} = 0$

and $(\vec{v} \cdot \vec{t}) \cdot (\vec{v} + \vec{t}) = (\vec{v} \cdot \vec{v}) - (\vec{t} \cdot \vec{t})$

$$= (v_1^2 + v_2^2) - (t_1^2 + t_2^2) = \|\vec{v}\|^2 - \|\vec{t}\|^2$$

53 For any vectors $\vec{v}$ and $\vec{t}$, $\vec{v} \cdot \vec{t} \leq \|\vec{v}\| \cdot \|\vec{t}\|$

By the triangle inequality, $\|\vec{v} + \vec{t}\| \leq \|\vec{v}\| + \|\vec{v}\|$

But $0 \leq \|\vec{v} + \vec{t}\|$

Then $\|\vec{v} + \vec{t}\|^2 \leq (\|\vec{v}\| + \|\vec{t}\|)^2$

$$(\vec{v} + \vec{t}) \cdot (\vec{v} + \vec{t}) \leq \|\vec{v}\|^2 + 2\|\vec{v}\|\|\vec{t}\| + \|\vec{t}\|^2$$

$$\vec{v} \cdot \vec{v} + 2(\vec{v} \cdot \vec{t}) + \vec{t} \cdot \vec{t}$$
$$\leq \vec{v} \cdot \vec{v} + 2\|\vec{v}\| \cdot \|\vec{t}\| + \vec{t} \cdot \vec{t}$$

$$\vec{v} \cdot \vec{t} \leq \|\vec{v}\| \cdot \|\vec{t}\|$$

(54) Show that the specified subsets of $\mathbb{R} \times \mathbb{R}$ are equal.

$$\{(x,y) : (x,y) = (2,3) + k(-1,1)\};$$

$$\{(x,y) : y = 5 - x\}$$

$(x = 2-k)$, $y = 3+k$

Eliminate k by writing k in terms of x.

$x = 2-k \leftrightarrow k = 2-x$

then substitute into $y = 3+k$

$y = 3 + (2-x) = 5-x$

which is the condition of the second set.

In other words, (x,y) belongs to the first subset when $(x,y) = (2,3) + (-k,k) = (2-k, 3+k)$, if and only if $x = 2-k$ and $y = 3+k$.

$k = 2-x$, $y = 3+(2-x) = 5-x$

Thus, any element of the first subset is an element of the second.

(x,y) belongs to the second subset when

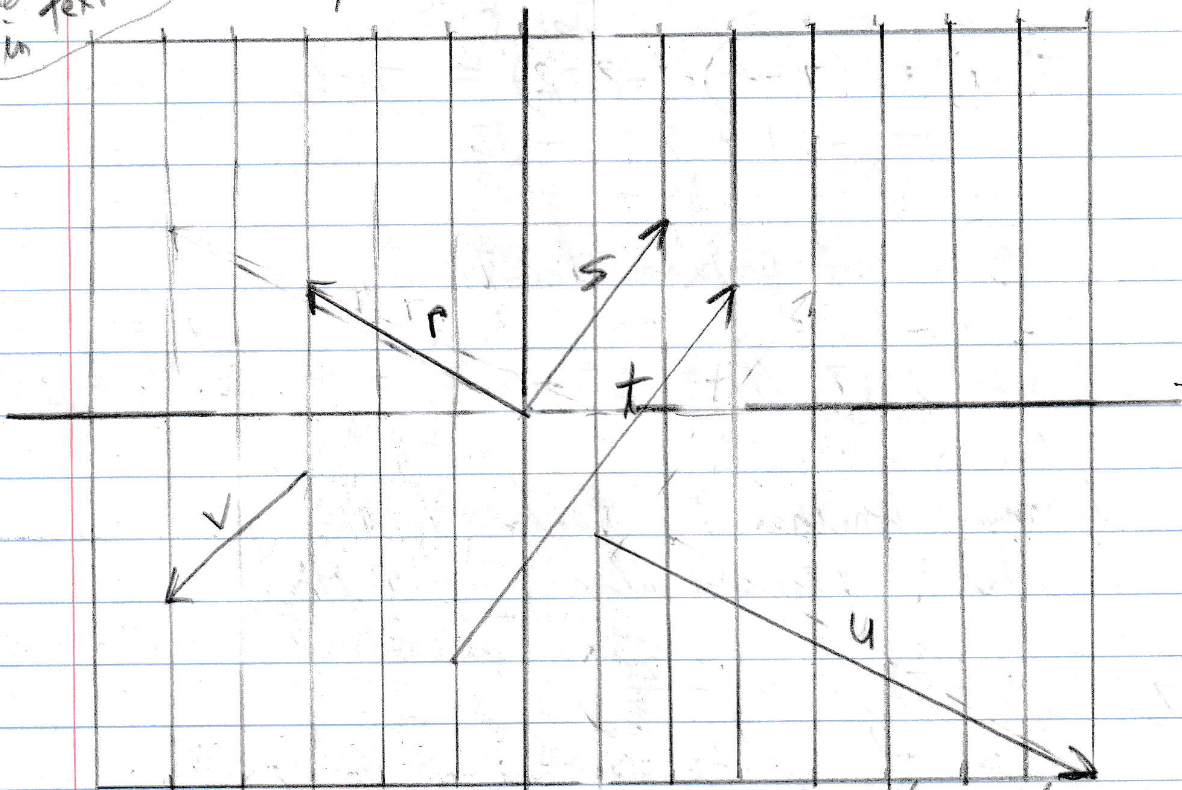
$(x,y) = (x, 5-x) = (x-2, (5-x)-3) + (2,3)$

$= (2,3) + (2-x)(1,-1)$; let $k = 2-x$ and any

element of the second subset is an element of the first

$\boxed{4\text{-}7}$ Relationships among Parallel and Perpendicular Vectors

Note error in text $\vec{r}$



Determine $\vec{r}, \vec{s}, \vec{t}, \vec{u}$, and $\vec{v}$. Verify each statement.

$\vec{r} = (-2,3)$; $\vec{s} = (2,3)$, $\vec{t} = (4,6)$, $\vec{v} = (7,-4)$,
$\vec{V} = (-2,-2)$

1. $\vec{r}$ is perpendicular to $\vec{s}$: $(-3,2)\cdot(2,3) = -6+6 = 0$
2. $\vec{s}$ is parallel to $\vec{t}$ (if and only if $\vec{s}\cdot\vec{t_p} = 0$)
   $\vec{s}\cdot\vec{t_p} = (2,3)\cdot(-6,4) = 0$ $-12+12 = 0$
3. $\vec{V}$ is not parallel to $\vec{t}$: $\vec{V}\cdot\vec{t_p} \neq 0$; $(-2,-2)\cdot(-6,4) = 12-8 = 4 \neq 0$

(4) $\vec{v}$ is not perpendicular to $\vec{u}$.

$$\vec{v} \cdot \vec{u} = (-2,-2)\cdot(7,-4) = (-2)\cdot 7 + (-2)(-4)$$
$$= -14 + 8 = -6 \neq 0$$

(5) $\vec{u}$ is not parallel to $\vec{r}$.

$$\vec{u} \cdot \vec{r_p} = (7,-4)\cdot(-3,-2) = 7(-3) + (-4)(-2)$$
$$= -21 + 8 = -13$$

(6) $\vec{s}$ is not perpendicular to $\vec{u}$.

$$\vec{s} \cdot \vec{u} = (2,3)\cdot(7,-4) = 2\cdot 7 + 3(-4) = 14 - 12 = 5 \neq 0$$

Determine whether the vectors in each pair are parallel, perpendicular, or neither.

(7) $(3,-5)$; $\left(4, -\frac{20}{3}\right)$

$$(3,-5)\cdot\left(\frac{20}{3}, 4\right) = 3\left(\frac{20}{3}\right) + (-5)(4)$$
$$= 20 - 20 = 0$$

$\therefore$ parallel

(8) $(3,7)$; $(7,-3)$

$$(3,7)\cdot(7,-3) = 3\cdot 7 + 7(-3) = 21 - 21 = 0$$

perpendicular

⑨ $(4,6)$; $(-2,3)$

$(4,6)\cdot(-3,-2) = 4(-3) + 6(-2) = -12 - 12 = -24 \neq 0$

$\therefore$ not parallel

$(4,6)\cdot(-2,3) = -8 + 18 = 10 \neq 0$ $\therefore$ not perpendicular

So, neither.

⑩ $(\sqrt{2}, \sqrt{3})$; $(\sqrt{6}, -2)$

$(\sqrt{2}, \sqrt{3})\cdot(\sqrt{6}, -2) = \sqrt{2}\cdot\sqrt{6} + \sqrt{3}(-2)$

$= \sqrt{12} - 2\sqrt{3} = 2\sqrt{3} - 2\sqrt{3} = 0$

$\therefore$ perpendicular

⑪ $\left(\frac{2}{3}, \frac{2}{3}\right)$; $(1,-1)$

$\left(\frac{2}{3}, \frac{2}{3}\right)\cdot(1,-1) = \frac{2}{3} - \frac{2}{3} = 0$

$\therefore$ perpendicular

⑫ $(\sqrt{3}, -2)$; $(6, -4\sqrt{3})$

$(\sqrt{3}, -2)\cdot(6, -4\sqrt{3}) = \sqrt{3}(6) + (-2)(-4\sqrt{3})$

$= 6\sqrt{3} + 8\sqrt{3} = 14\sqrt{3} \neq 0$ $\therefore$ not perpendicular

$(\sqrt{3}, -2)\cdot(4\sqrt{3}, 6) = \sqrt{3}(4\sqrt{3}) + (-2)(6)$

$= 12 - 12 = 0$ $\therefore$ parallel

Find the value of k for which each pair of vectors is (a) perpendicular, (b) parallel

In exercises 17-18 assume no vector entry is 0.

13 $(3,-5)$; $(5,k)$

(a) $(3,-5)\cdot(5,k)=0$

$3\cdot 5-5k=0 \leftrightarrow 15=5k \leftrightarrow k=3$

(b) $(3,-5)\cdot(-k,5)=0$

$-3k-25=0 \leftrightarrow -3k=25 \leftrightarrow k=-\frac{25}{3}$

14 $(k,5)$; $(\sqrt{3},3)$

(a) $(k,5)\cdot(\sqrt{3},3)=0$

$\sqrt{3}k+5\cdot 3=0 \leftrightarrow \sqrt{3}k=-15 \leftrightarrow k=-\frac{15}{\sqrt{3}}$

$k=-\frac{15}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}=-\frac{15\sqrt{3}}{3}=-5\sqrt{3}$

(b) $(k,5)\cdot(-3,\sqrt{3})=0$

$-3k+5\sqrt{3}=0 \leftrightarrow 3k=5\sqrt{3}$

$k=\frac{5}{3}\sqrt{3}$

$\textcircled{15}$ $(k,-1);\ (5,-10)$

(a) $(k,-1)\cdot(5,-10)=0$

$k(5)+(-1)(-10) = 5k+10=0 \leftrightarrow 5k=-10$

$k=-2$

(b) $(k,-1)\cdot(10,5)=0$

$10k-5=0 \leftrightarrow 10k=5 \leftrightarrow k=\frac{1}{2}$

$\textcircled{16}$ $(3,\sqrt{2});\ (-\sqrt{2},k)$

(a) $(3,\sqrt{2})\cdot(-\sqrt{2},k)=0$

$3(-\sqrt{2})+\sqrt{2}k=0 \leftrightarrow -3\sqrt{2}+\sqrt{2}k=0$

$\sqrt{2}k=3\sqrt{2} \leftrightarrow k=3$

(b) $(3,\sqrt{2})\cdot(-k,-\sqrt{2})=0$

$3(-k)+\sqrt{2}(-\sqrt{2})=0 \leftrightarrow -3k-2=0$

$-3k=2 \leftrightarrow k=-\frac{2}{3}$

$\textcircled{17}$ $(r,s);\ (-s,k)$

(a) $(r,s)\cdot(-s,k)=0$

$r(-s)+sk=0 \leftrightarrow sk=rs \leftrightarrow k=r$

(b) $(r,s)\cdot(-k,-s)=0$

$r(-k)+s(-s)=0 \leftrightarrow -r\cdot k=s^2$

$k=-\frac{s^2}{r}$

(18) (a, k); (c, d)

(a) $(a, k) \cdot (c, d) = 0$

$a \cdot c + kd = 0 \leftrightarrow kd = -ac$

$k = -\frac{ac}{d}$

(b) $(a, k) \cdot (-d, c) = 0$

$a(-d) + kc = 0 \leftrightarrow k \cdot c = ad$

$k = \frac{ad}{c}$

(19) Given $\vec{v} = (2, 1) + (4, k)$ and

$\vec{s} = (2, -6) + (k, -2)$

Determine the value of k for which $\vec{v}$ is perpendicular to $\vec{s}$.

$\vec{v} = (6, k+1)$; $\vec{s} = (k+2, -8)$

$\vec{v} \cdot \vec{s} = 0$ when $(6, k+1) \cdot (k+2, -8) = 0$

$\leftrightarrow \quad 6(k+2) + -8(k+1) = 0$

$6k + 12 - 8k - 8 = 0 \leftrightarrow -2k + 4 = 0$

$-2k = -4 \leftrightarrow k = 2$

(20)

(21)

(22)

(20) Given $\vec{r} = (6, 4\sqrt{2})$ and $\vec{s} = (-2, 8)$.
Show that $\vec{r} + \vec{s}$ is perpendicular to $\vec{r} - \vec{s}$.

$$\vec{r}+\vec{s} = (4, 4\sqrt{2}+8)$$
$$\vec{r}-\vec{s} = (8, 4\sqrt{2}-8)$$

$$(\vec{r}+\vec{s})\cdot(\vec{r}-\vec{s}) = 0$$
$$(4, 4\sqrt{2}+8)\cdot(8, 4\sqrt{2}-8) = 0$$
$$4\cdot 8 + (4\sqrt{2}+8)(4\sqrt{2}-8) = 0$$
$$32 + 32 - 64 = 0$$

(21) Prove: No vector other than $(0,0)$ can be perpendicular to itself.

Note: $\|\vec{v}\| \geq 0$
$\|\vec{v}\| = 0$ if and only if $\vec{v} = \vec{0}$

$$\vec{v}\cdot\vec{v} = (v_1, v_2)\cdot(v_1, v_2) = v_1^2 + v_2^2 = \|\vec{v}\|$$

If $\vec{v}$ is perpendicular to itself, then $\vec{v}\cdot\vec{v} = 0$,
$v_1^2 + v_2^2 = 0$, $\|\vec{v}\|^2 = 0$
This is only true when $\vec{v} = \vec{0} = (0,0)$

(22) Prove: $\vec{0}$ is perpendicular to every vector.
$\vec{v} = (v_1, v_2)$, $v_1, v_2 \in R$, $\vec{v}\cdot\vec{0} = v_1(0) + v_2(0) = 0$
$\therefore$ $\vec{0}$ is perpendicular to any vector $\vec{v}$.

(23) Prove: If a nonzero vector $\vec{s}$ is perpendicular to $\vec{t}$, $-\vec{s}$ is perpendicular to $\vec{t}$.

$(s_1, s_2) \cdot (t_1, t_2) = 0$

$s_1 t_1 + s_2 t_2 = 0$

$s_1 t_1 = -s_2 t_2$

$(-\vec{s}) \cdot \vec{t} = 0 \leftrightarrow (-s_1, -s_2) \cdot (t_1, t_2) = 0$

$\leftrightarrow -s_1 t_1 - s_2 t_2 = 0 \leftrightarrow -s_2 t_2 = s_1 t_1$

$\therefore$ if $\vec{s}$ is perpendicular to $\vec{t}$ then $-\vec{s}$ is perpendicular to $\vec{t}$.

(24) Prove: If $\vec{v}$ is perpendicular to $\vec{s}$, then, for any scalars k and r, $k\vec{v}$ is perpendicular to $r\vec{s}$.

If $\vec{v} \cdot \vec{s} = 0$, then $(k\vec{v}) \cdot (r\vec{s}) = 0$

Proof: $k(v_1, v_2) \cdot r(s_1, s_2) = (kv_1, kv_2) \cdot (rs_1, rs_2)$

$= krv_1 s_1 + krv_2 s_2 = kr(v_1 s_1 + v_2 s_2)$

$= k \cdot r (\vec{v} \cdot \vec{s}) = k \cdot r \cdot (0) = 0$

$\therefore$ $k\vec{v}$ is perpendicular to $r\vec{s}$.

(25)

(26)

(25) Given: $\vec{v} \neq \vec{0}$; $M = \{\text{vectors perpendicular to } r\vec{v},\ r \neq 0\}$; $S = \{\text{vectors parallel to } \vec{v}_p\}$. Prove $M = S$

$$\vec{v} = (v_1, v_2) \neq \vec{0}$$

$$M = \{(x,y) : (x,y)\cdot(r\cdot v_1, r\cdot v_2) = 0,\ r \neq 0\}$$

$$= \{(x,y) : r\cdot x\cdot v_1 + r\cdot y\cdot v_2 = 0\}$$

$$= \{(x,y) : x = -\frac{v_2}{v_1} y\}$$

$$S = \{(x,y) : (x,y) \text{ is perpendicular to } (\vec{v}_p)_p = (-v_2, v_1)_p = (-v_1, -v_2)\}$$

$$= \{(x,y) : (x,y)\cdot(-v_1, -v_2) = 0\}$$

$$= \{(x,y) : -xv_1 - yv_2 = 0\} \qquad x = -\frac{v_2}{v_1} y$$

$$= \{(x,y) : x = -\frac{v_2}{v_1} y\} \quad \therefore\ S = M$$

(26) Prove: $(\vec{v}_p)_p = -\vec{v}$

$$[(v_1, v_2)_p]_p = (-v_2, v_1)_p = (-v_1, -v_2) = -\vec{v}$$

$\boxed{4\text{-}8}$ Perpendicular Components of Vectors

For each given vector $\vec{v}$ determine a unit vector in the same direction.

① $\vec{v} = (3,-2)$; $\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|}$

$$\vec{u} = \frac{(3,-2)}{\sqrt{3^2+(-2)^2}} = \frac{(3,-2)}{\sqrt{13}} = \left(\frac{3\sqrt{13}}{13}, \frac{-2\sqrt{13}}{13}\right)$$

② $\vec{v} = (-2,5)$

$$\vec{u} = \frac{(-2,5)}{\sqrt{(-2)^2+5^2}} = \frac{(-2,5)}{\sqrt{29}} = \left(\frac{-2}{\sqrt{29}}, \frac{5}{\sqrt{29}}\right)$$

③ $\vec{v} = (a,-b)$, $\vec{v} \neq \vec{0}$

$$\vec{u} = \frac{(a,-b)}{\sqrt{a^2+b^2}} = \left(\frac{a}{\sqrt{a^2+b^2}}, \frac{-b}{\sqrt{a^2+b^2}}\right)$$

④ $\vec{v} = (3+\sqrt{2}, 3-\sqrt{2})$

$$\vec{u} = \frac{(3+\sqrt{2}, 3-\sqrt{2})}{\sqrt{(3+\sqrt{2})^2+(3-\sqrt{2})^2}} = \frac{(3+\sqrt{2}, 3-\sqrt{2})}{\sqrt{(11+6\sqrt{2})+(11-6\sqrt{2})}}$$

$$= \frac{(3+\sqrt{2}, 3-\sqrt{2})}{\sqrt{22}} = \left(\frac{3+\sqrt{2}}{\sqrt{22}}, \frac{3-\sqrt{2}}{\sqrt{22}}\right)$$

Which of the following are unit vectors?

(5) $\left(4, \frac{1}{2}\right)$: $4^2 + \left(\frac{1}{2}\right)^2 = 16\frac{1}{4} \neq 1$

$\therefore$ not a unit vector

(6) $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$: $\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = 1$

$\therefore$ unit vector

(7) $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$: $\left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + \frac{1}{2} = 1$

$\therefore$ unit vector

(8) $\left(\frac{1-\sqrt{2}}{2}, \frac{1+\sqrt{2}}{2}\right)$: $\left(\frac{1-\sqrt{2}}{2}\right)^2 + \left(\frac{1+\sqrt{2}}{2}\right)^2 =$

$\frac{1-2\sqrt{2}+2}{4} + \frac{1+2\sqrt{2}+2}{4} = \frac{6}{4} = \frac{3}{2} \neq 1$ $\therefore$ not unit vector

For what values of k is each of the following a unit vector?

(9) $\left(\frac{3}{k}, \frac{4}{k}\right)$: $\left(\frac{3}{k}\right)^2 + \left(\frac{4}{k}\right)^2 = 1$

$\frac{9}{k^2} + \frac{16}{k^2} = 1 \longleftrightarrow 9 + 16 = k^2 \longleftrightarrow k = \pm 5$

(10) $\left(\frac{1}{3}, \frac{k}{3}\right)$: $\left(\frac{1}{3}\right)^2 + \left(\frac{k}{3}\right)^2 = 1 \longleftrightarrow \frac{1}{9} + \frac{k^2}{9} = 1$

$1 + k^2 = 9 \longleftrightarrow k^2 = 8 \longleftrightarrow k = \pm 2\sqrt{2}$

(11) $\left(\frac{\sqrt{2}}{k}, \frac{\sqrt{7}}{k}\right)$; $\left(\frac{\sqrt{2}}{k}\right)^2 + \left(\frac{\sqrt{7}}{k}\right)^2 = 1$

$\longleftrightarrow \quad \frac{2}{k^2} + \frac{7}{k^2} = 1 \longleftrightarrow 2 + 7 = k^2 \longleftrightarrow k^2 = 9$

$\longleftrightarrow \quad k = \pm 3$

(12) $\left(\frac{2}{5}, \frac{k}{10}\right)$; $\left(\frac{2}{5}\right)^2 + \left(\frac{k}{10}\right)^2 = 1$

$\longleftrightarrow \quad \frac{4}{25} + \frac{k^2}{100} = 1 \longleftrightarrow \frac{100(4)}{25} + k^2 = 100$

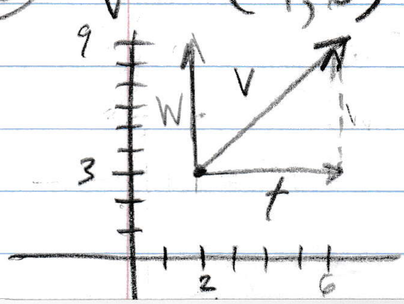
$\longleftrightarrow \quad 16 + k^2 = 100 \longleftrightarrow k^2 = 84 \longleftrightarrow k = \pm\sqrt{84}$

$= \pm 2\sqrt{21}$

Represent the vector $\vec{v}$ as an arrow in a coordinate plane with initial point at $(2,3)$. Draw and give a formula for the resolution of $\vec{v}$ into two perpendicular components $\vec{w}$ and $\vec{t}$ such that $\vec{t}$ is parallel to the x-axis.

Represent $\vec{w}$ and $\vec{t}$ as arrows with the same initial point as $\vec{v}$ and indicate the norm of each.

(13) $\vec{v} = (4,6)$



$\vec{t} = (4,0)$; $\|\vec{t}\| = \sqrt{4^2+0^2} = 4$

$\vec{w} = (0,6)$; $\|\vec{w}\| = \sqrt{6^2} = 6$

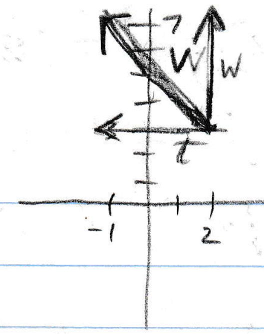
(14) $\vec{v} = (-3, 4)$

$\vec{w} = (0, 4)$

$\vec{t} = (-3, 0)$

$\|\vec{w}\| = 4$

$\|\vec{t}\| = 3$

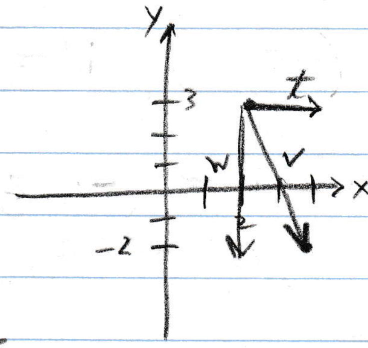


(15) $\vec{v} = (2, -5)$

$\vec{t} = (2, 0)$

$\vec{w} = (0, -5)$

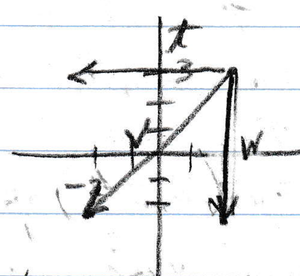
$\|\vec{t}\| = 2$, $\|\vec{w}\| = 5$



(16) $\vec{v} = (-4, -5)$

$\vec{t} = (-4, 0)$, $\|\vec{t}\| = 4$

$\vec{w} = (0, -5)$, $\|\vec{w}\| = 5$



Given $\vec{w} + \vec{t} = \vec{v}$, In each exercise find $\vec{t}$ and tell whether $\vec{w}$ and $\vec{t}$ are perpendicular components of $\vec{v}$.

(17) $\vec{w} = (-3, 5)$, $\vec{v} = (8, -2)$

$\vec{t} = \vec{v} - \vec{w} = (8+3, -2-5) = (11, -7)$

$\vec{t} \cdot \vec{w} = (11, -7) \cdot (-3, 5) = -33 - 35 = -68 \neq 0$

$\therefore \vec{t}$ is not perpendicular to $\vec{w}$.

(18) $\vec{w} = (5,3)$; $\vec{v} = (8,8)$ $\vec{w} + \vec{t} = \vec{v}$

$\vec{t} = \vec{v} - \vec{w} = (8,8) - (5,3) = (3,5)$

$\vec{t} \cdot \vec{w} = (3,5) \cdot (5,3) = 15 + 15 = 30 \neq 0$

$\therefore$ $\vec{t}$ is not perpendicular to $\vec{w}$

(19) $\vec{w} = (-5,10)$; $\vec{v} = (-1,12)$

$\vec{t} = \vec{v} - \vec{w} = (-1,12) - (-5,10)$
$= (-1,12) + (5,-10) = (4,2)$

$\vec{t} \cdot \vec{w} = (4,2) \cdot (-5,10) = -20 + 20 = 0$

$\therefore$ $\vec{t}$ and $\vec{w}$ are perpendicular components of $\vec{v}$

(20) $\vec{w} = (2,-4)$; $\vec{v} = (10,8)$

$\vec{t} = \vec{v} - \vec{w} = (10,8) - (2,-4)$
$= (10,8) + (-2,4) = (8,12)$

$\vec{t} \cdot \vec{w} = (8,12) \cdot (2,-4) = 16 - 48 = -32$

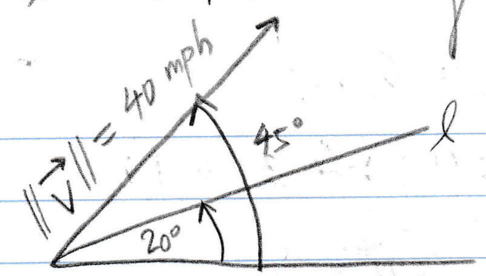
$-32 \neq 0$ $\therefore$ $\vec{t}$ is not perpendicular to $\vec{w}$.

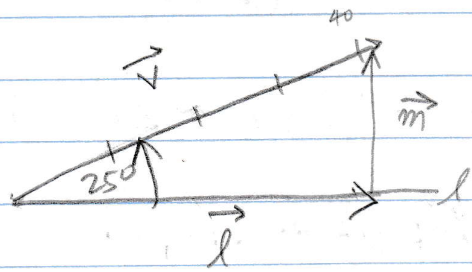
In exercises 21 & 22 make a scale drawing of the given figure. Then graphically resolve $\vec{v}$ into two perpendicular component vectors, one of which lies along l.

Approximate the norm of each component vector.

$\vec{v} = (40\cos(25^\circ), 40\sin(25^\circ))$
$= (36.2523, 16.9)$

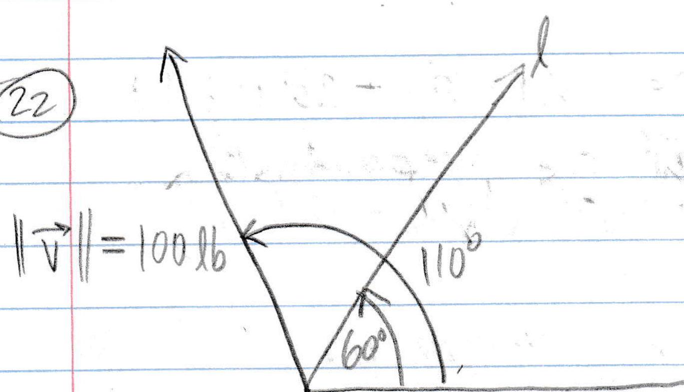
(21)

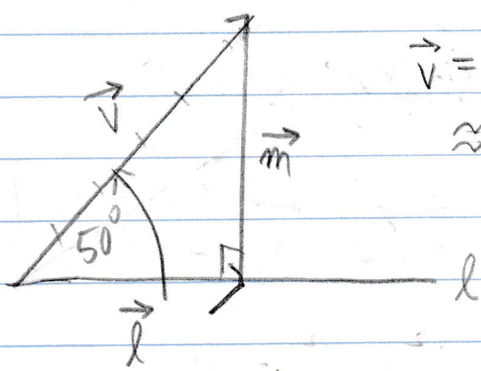




$\|\vec{l}\| \approx 36$
$\|\vec{m}\| \approx 17$

(22)





$\vec{v} = (100\cos(50^\circ), 100\sin(50^\circ))$
$\approx (64.2788, 76.6)$

$\|\vec{l}\| \approx 64$

$\|\vec{m}\| \approx 77$

Express $\vec{v}$ as a linear combination of $\vec{u} = (1,0)$ and $\vec{u}_p = (0,1)$.

(23) $\vec{v} = (5,-2) = 5(1,0) - 2(0,1)$

(24) $\vec{v} = (3,-1) = 3(1,0) - 1(0,1)$

(25) $\vec{v} = (-4,-2) = -4(1,0) - 2(0,1)$

(26) $\vec{v} = (0,5) = 0(1,0) + 5(0,1)$

(27) $\vec{v} = (-2, \sqrt{5}) = -2(1,0) + \sqrt{5}(0,1)$

(28) $\vec{v} = (\sqrt{3},-2) = \sqrt{3}(1,0) - 2(0,1)$

For what values of k will the given vector be a unit vector?

(29) $\left(\frac{1}{k}, 0\right)$; $\left(\frac{1}{k}\right)^2 + 0^2 = 1$

$$\frac{1}{k^2} = 1 \longleftrightarrow 1 = k^2 \longleftrightarrow k = \pm 1$$

(30) $(k-1, k)$: $(k-1)^2 + k^2 = 1$

$\longleftrightarrow$ $k^2 - 2k + 1 + k^2 = 1 \longleftrightarrow 2k^2 - 2k + 1 = 1$

$\longleftrightarrow$ $2k^2 - 2k = 0 \longleftrightarrow 2k(k-1) = 0$

$k = 0$ or $k = 1$

(31) $\left(\frac{k}{5}, \frac{k+1}{5}\right)$: $\left(\frac{k}{5}\right)^2 + \left(\frac{k+1}{5}\right)^2 = 1$

$$\longleftrightarrow \frac{k^2}{25} + \frac{k^2+2k+1}{25} = 1 \longleftrightarrow k^2 + k^2 + 2k + 1 = 25$$

$$\longleftrightarrow 2k^2 + 2k - 24 = 0 \longleftrightarrow k^2 + k - 12 = 0$$
$$\longleftrightarrow (k+4)(k-3) = 0 \quad k = -4 \text{ or } k = 3$$

(32) $\left(\frac{k-2}{13}, \frac{k+5}{13}\right)$: $\left(\frac{k-2}{13}\right)^2 + \left(\frac{k+5}{13}\right)^2 = 1$

$$\longleftrightarrow \frac{k^2-4k+4}{169} + \frac{k^2+10k+25}{169} = 1$$

$$\longleftrightarrow k^2 - 4k + 4 + k^2 + 10k + 25 = 169$$
$$\longleftrightarrow 2k^2 + 6k - 140 = 0 \longleftrightarrow k^2 + 3k - 70 = 0$$
$$\longleftrightarrow (k+10)(k-7) = 0 \longleftrightarrow k = -10 \text{ or } k = 7$$

In exercise 33–38 determine the components of $\vec{v}$ parallel to and perpendicular to $\vec{w}$. In each exercise show that the sum of the components is $\vec{v}$.

(33) $\vec{v} = (5,3)$, $\vec{w} = (1,-3)$

$$\overrightarrow{Comp_w}\,\vec{v} = \frac{\vec{w}\cdot\vec{v}}{\|\vec{w}\|^2}\vec{w} = \frac{(1,-3)\cdot(5,3)}{(1,-3)\cdot(1,-3)}(1,-3)$$

$$= \frac{5-9}{1+9}(1,-3) = -\frac{4}{10}(1,-3) = -\frac{2}{5}(1,-3) = \left(-\frac{2}{5}, \frac{6}{5}\right)$$

I used $\|\vec{w}\|^2 = \vec{w}\cdot\vec{w}$ since $\left(\sqrt{w_1^2 + w_2^2}\right)^2 = w_1^2 + w_2^2$

$$\left[\frac{(1,-3)\cdot(5,3)}{\left(\sqrt{1^2+(-3)^2}\right)^2} = \frac{-4}{10}\right]$$

So, $\overrightarrow{\text{Comp}}_W \vec{v} = \left(-\frac{2}{5}, \frac{6}{5}\right)$ $\quad \vec{v} = (5,3)$; $\vec{w} = (1,-3)$

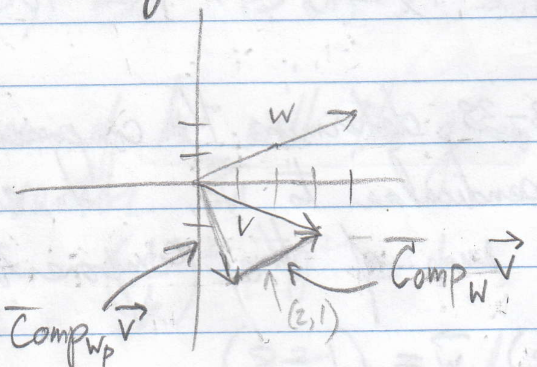
$$\overrightarrow{\text{Comp}}_{W_p} \vec{v} = \frac{\vec{W_p} \cdot \vec{v}}{\|\vec{w}\|^2} \vec{W_p} = \frac{(3,1)\cdot(5,3)}{(1,-3)\cdot(1,-3)}(3,1)$$

$$= \frac{15+3}{10}(3,1) = \frac{9}{5}(3,1) = \left(\frac{27}{5}, \frac{9}{5}\right)$$

$$\overrightarrow{\text{Comp}}_W \vec{v} + \overrightarrow{\text{Comp}}_{W_p} \vec{v} = \left(-\frac{2}{5}, \frac{6}{5}\right) + \left(\frac{27}{5}, \frac{9}{5}\right) = (5,3) = \vec{v}$$

(34) $\vec{v} = (3,-1)$; $\vec{w} = (4,2)$

* Determine the components of $\vec{v}$ parallel and perpendicular to $\vec{w}$. Let's imagine this geometrically before solving algebraically



$$\overrightarrow{\text{Comp}}_W \vec{v} = \frac{\vec{w}\cdot\vec{v}}{\vec{w}\cdot\vec{w}}\vec{w} = \frac{(4,2)\cdot(3,-1)}{(4,2)\cdot(4,2)}(4,2) = \frac{10}{20}(4,2)$$

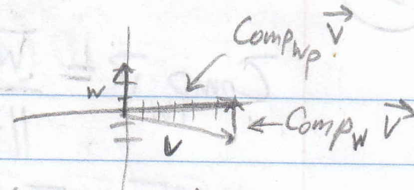
$$= (2,1)$$

$\vec{W_p} = (-2,4)$

$$\overrightarrow{\text{Comp}}_{W_p} \vec{v} = \frac{\vec{W_p}\cdot\vec{v}}{\vec{w}\cdot\vec{w}}\vec{W_p} = \frac{(-2,4)\cdot(3,-1)}{(4,2)\cdot(4,2)}(-2,4)$$

$$= \frac{-10}{20}(-2,4) = (1,-2)$$

$$\overline{Comp}_W \vec{v} + \overline{Comp}_{W_p} \vec{v} = (2,1) + (1,-2) = (3,-1) = \vec{v}$$



(35) $\vec{v} = (6,-2)$; $\vec{W} = (0,3)$

$$\overline{Comp}_W \vec{v} = \frac{\vec{W} \cdot \vec{v}}{\|\vec{W}\|^2} \vec{W} = \frac{(0,3) \cdot (6,-2)}{(\sqrt{3^2+0^2})^2}(0,3) = \frac{-6}{9}(0,3)$$

$$= (0,-2)$$

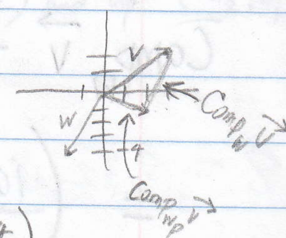
$\vec{W_p} = (-3,0)$

$$\overline{Comp}_{W_p} \vec{v} = \frac{\vec{W_p} \cdot \vec{v}}{\|W^2\|} \vec{W_p} = \frac{(-3,0)\cdot(6,-2)}{9}(-3,0) = \frac{-18}{9}(-3,0)$$

$$= (6,0)$$

$$\overline{Comp}_W \vec{v} + \overline{Comp}_{W_p} \vec{v} = (0,-2) + (6,0) = (6,-2) = \vec{v}$$

(36) $\vec{v} = (3,2)$; $\vec{W} = (-1,-4)$ $\quad \vec{W_p} = (4,-1)$



$$\overline{Comp}_W \vec{v} = \frac{\vec{W} \cdot \vec{v}}{\vec{W} \cdot \vec{W}} \vec{W} = \frac{(-1,-4)\cdot(3,2)}{(-1,-4)\cdot(-1,-4)}(-1,-4)$$

$$= \frac{-3-8}{1+16}(-1,-4) = -\frac{11}{17}(-1,-4) = \left(\frac{11}{17}, \frac{44}{17}\right)$$

$$\overline{Comp}_{W_p} \vec{v} = \frac{\vec{W_p} \cdot \vec{v}}{\vec{W} \cdot \vec{W}} \vec{W_p} = \frac{(4,-1)\cdot(3,2)}{17}(4,-1) = \frac{10}{17}(4,-1)$$

$$= \left(\frac{40}{17}, -\frac{10}{17}\right)$$

$$\overline{Comp}_W \vec{v} + \overline{Comp}_{W_p} \vec{v} = \left(\frac{11}{17}, \frac{44}{17}\right) + \left(\frac{40}{17}, \frac{-10}{17}\right) = \left(\frac{51}{17}, \frac{34}{17}\right)$$

$$= (3,2) = \vec{v}$$

(37) $\vec{v} = (2\sqrt{3}, \sqrt{3})$; $\vec{w} = (2,-1)$ so $\vec{w_p} = (1,2)$

$$\text{Comp}_{w}\vec{v} = \frac{\vec{w}\cdot\vec{v}}{\|\vec{w}\|^2}\vec{w} = \frac{(2,-1)\cdot(2\sqrt{3},\sqrt{3})}{\left(\sqrt{2^2+(-1)^2}\right)^2}(2,-1)$$

$$= \frac{4\sqrt{3}-\sqrt{3}}{5}(2,-1) = \frac{3\sqrt{3}}{5}(2,-1) = \left(\frac{6\sqrt{3}}{5}, -\frac{3\sqrt{3}}{5}\right)$$

$$\text{Comp}_{w_p}\vec{v} = \frac{\vec{w_p}\cdot\vec{v}}{\|\vec{w}\|^2}\vec{w_p} = \frac{(1,2)\cdot(2\sqrt{3},\sqrt{3})}{5}(1,2)$$

$$= \frac{4\sqrt{3}}{5}(1,2) = \left(\frac{4\sqrt{3}}{5}, \frac{8\sqrt{3}}{5}\right)$$

$$\text{Comp}_{w}\vec{v} + \text{Comp}_{w_p}\vec{v} = \left(\frac{6\sqrt{3}}{5}, -\frac{3\sqrt{3}}{5}\right) + \left(\frac{4\sqrt{3}}{5}, \frac{8\sqrt{3}}{5}\right)$$

$$= \left(\frac{10\sqrt{3}}{5}, \frac{5\sqrt{3}}{5}\right) = (2\sqrt{3}, \sqrt{3}) = \vec{v}$$

(38) $\vec{v} = (4, \sqrt{3})$; $\vec{w} = (\sqrt{3}, -1)$ so $\vec{w_p} = (1, \sqrt{3})$

$$\text{Comp}_{w}\vec{v} = \frac{\vec{w}\cdot\vec{v}}{\vec{w}\cdot\vec{w}}\vec{w} = \frac{(\sqrt{3},-1)\cdot(4,\sqrt{3})}{(\sqrt{3},-1)\cdot(\sqrt{3},-1)}(\sqrt{3},-1)$$

$$= \frac{4\sqrt{3}-\sqrt{3}}{3+1}(\sqrt{3},-1) = \frac{3\sqrt{3}}{4}(\sqrt{3},-1) = \left(\frac{9}{4}, \frac{-3\sqrt{3}}{4}\right)$$

$$\overline{\text{Comp}}_{W_p}\vec{V} = \frac{\vec{W_p}\cdot\vec{V}}{\vec{W}\cdot\vec{W}}\vec{W_p} = \frac{(1,\sqrt{3})\cdot(4,\sqrt{3})}{4}(1,\sqrt{3})$$

$$= \frac{7}{4}(1,\sqrt{3}) = \left(\frac{7}{4}, \frac{7\sqrt{3}}{4}\right)$$

$$\overline{\text{Comp}}_{W}\vec{V} + \overline{\text{Comp}}_{W_p}\vec{V} = \left(\frac{9}{4}, -\frac{3\sqrt{3}}{4}\right) + \left(\frac{7}{4}, \frac{7\sqrt{3}}{4}\right)$$

$$= \left(\frac{16}{4}, \frac{4\sqrt{3}}{4}\right) = (4, \sqrt{3}) = \vec{V}$$

Find the unit vector $\vec{u}$ such that

(39) $(2,11) = 10\vec{u} + 5\vec{u_p}$

$10 = \vec{u}\cdot\vec{V}$, $5 = \vec{u_p}\cdot\vec{V}$ where $\vec{u} = (u_1, u_2)$ and $\vec{u_p} = (-u_2, u_1)$

$\vec{V} = (2, 11)$

$10 = (u_1, u_2)\cdot(2,11) = 2u_1 + 11u_2$

$5 = (-u_2, u_1)\cdot(2,11) = -2u_2 + 11u_1$

$$\begin{aligned} 2u_1 + 11u_2 &= 10 \\ 11u_1 - 2u_2 &= 5 \end{aligned} \qquad \begin{bmatrix} 2 & 11 & 10 \\ 11 & -2 & 5 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 3/5 \\ 0 & 1 & 4/5 \end{bmatrix}$$

$\vec{u} = \left(\frac{3}{5}, \frac{4}{5}\right)$ I used "Linear Algebra". How would we proceed in a different manner?

$(2,11) = 10\vec{u} + 5\vec{u_p} = (10\cdot u_1, 10u_2) + (-5u_2, 5u_1)$

$(2,11) = (10u_1 - 5u_2, 10u_2 + 5u_1)$

$2 = 10u_1 - 5u_2$; $11 = 10u_2 + 5u_1$

$5u_2 = 10u_1 - 2$, $11 = 2(10u_1 - 2) + 5u_1 = 25u_1 - 4$

$u_2 = \frac{1}{5}(10u_1 - 2)$, $25u_1 = 15 \longleftrightarrow u_1 = \frac{15}{25} = \frac{3}{5}$

Since $u_1 = \frac{3}{5}$

$$u_2 = \frac{1}{5}\left(10\left(\frac{3}{5}\right) - 2\right) = \frac{1}{5}(6-2) = \frac{4}{5}$$

$$\therefore \vec{u} = \left(\frac{3}{5}, \frac{4}{5}\right)$$

Same results. I would enjoy showing both methods for posterity.

(40) $(8,2) = 3\sqrt{2}\,\vec{u} + 5\sqrt{2}\,\vec{u}_p$

$$(8,2) = 3\sqrt{2}\,\vec{u} + 5\sqrt{2}\,\vec{u}_p$$
$$= (3\sqrt{2}u_1, 3\sqrt{2}u_2) + (-5\sqrt{2}u_2, 5\sqrt{2}u_1)$$
$$= (3\sqrt{2}u_1 - 5\sqrt{2}u_2, 3\sqrt{2}u_2 + 5\sqrt{2}u_1)$$

$$8 = 3\sqrt{2}u_1 - 5\sqrt{2}u_2$$
$$3\sqrt{2}u_1 = 8 + 5\sqrt{2}u_2$$
$$u_1 = \frac{8 + 5\sqrt{2}u_2}{3\sqrt{2}}$$

$$2 = 3\sqrt{2}u_2 + 5\sqrt{2}u_1$$
$$3\sqrt{2}u_2 = 2 - 5\sqrt{2}\left(\frac{8 + 5\sqrt{2}u_2}{3\sqrt{2}}\right)$$
$$3\sqrt{2}u_2 = 2 - \frac{5}{3}(8 + 5\sqrt{2}u_2)$$

$$3\sqrt{2}u_2 = 2 - \frac{40}{3} - \frac{25\sqrt{2}}{3}u_2 = \frac{-34}{3} - \frac{25\sqrt{2}}{3}u_2$$

$$\left(\frac{9\sqrt{2} + 25\sqrt{2}}{3}\right)u_2 = \frac{-34}{3} \leftrightarrow 34\sqrt{2}u_2 = -34$$

$$u_2 = -\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = -\frac{\sqrt{2}}{2}; \quad u_1 = \frac{8 + 5\sqrt{2}\left(-\frac{\sqrt{2}}{2}\right)}{3\sqrt{2}} = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\vec{u} = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$$

Is my method any easier? Well, if I use a computer algebra system to put augmented matrix in row-reduced echelon form, yes.

$(8,2) = 3\sqrt{2}\,\vec{u} + 5\sqrt{2}\,\vec{u}_p$ $\qquad \vec{V} = a\vec{u} + b\vec{u}_p$

where $a = \vec{u}\cdot\vec{V}$ and $b = \vec{u}_p\cdot\vec{V}$

$3\sqrt{2} = (u_1, u_2)\cdot(8,2) = 8u_1 + 2u_2$

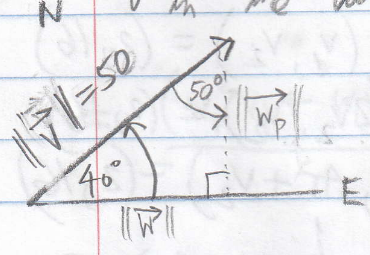
$5\sqrt{2} = (-u_2, u_1)\cdot(8,2) = -8u_2 + 2u_1 = 2u_1 - 8u_2$

Augmented matrix $m = \begin{bmatrix} 8 & 2 & 3\sqrt{2} \\ 2 & -8 & 5\sqrt{2} \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & -\frac{\sqrt{2}}{2} \end{bmatrix}$

$\vec{u} = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

In exercises 41-42 state approximations on the basis of a scale drawing.

41 A force of 50 pounds is applied in a direction 40° north of east. Find the scalar components of this force in the directions east and north.



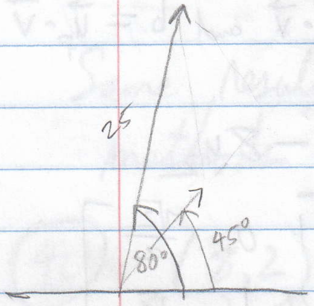
$\cos(40°) = \frac{\|\vec{W}\|}{50} \longleftrightarrow \|\vec{W}\| = 50\cos(40°) \approx 38$

$\sin(40°) = \frac{\|\vec{W}_p\|}{50} \longleftrightarrow \|\vec{W}_p\| \approx 50\sin(40°) \approx 32$

$\vec{V} = (50\cos(40°), 50\sin(40°))$

$\approx (38.3, 32.139)$

(42) A ship heads in the direction 80° at 25 knots. The tide is running at 5 knots in the direction 45°. Find the resultant speed and direction of the ship.

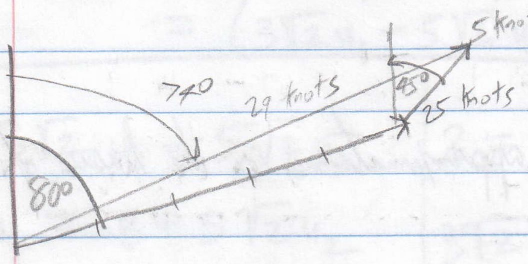


$$\left(25\cos(80^\circ), 25\sin(80^\circ)\right) + \left(5\cos(45^\circ), 5\sin(45^\circ)\right)$$

$$\vec{r} \approx (7{,}87,\ 28{,}15)$$

$$\|\vec{r}\| \approx \sqrt{(7{,}87)^2 + (28{,}15)^2} \approx 29{,}23 \approx 29 \text{ knots}$$

$$\alpha = \arctan\left(\frac{28{,}15}{7{,}87}\right) \approx 1{,}298 \text{ radians} \approx 74{,}38^\circ \approx 74^\circ$$



(43) Find a scalar r and a vector $\vec{v}$ such that $\vec{v}$ is perpendicular to $(-2, 4)$ and $r(-2,4) + \vec{v} = (2, 16)$

$$(v_1, v_2) \cdot (-2, 4) = 0$$
$$-2v_1 + 4v_2 = 0$$
$$4v_2 = 2v_1$$
$$v_1 = 2v_2$$

$$(-2r, 4r) + (v_1, v_2) = (2, 16)$$
$$(-2r, 4r) + (2v_2, v_2) = (2, 16)$$
$$(-2r + 2v_2,\ 4r + v_2) = (2, 16)$$

$$2 = -2r + 2v_2$$
$$16 = 4r + v_2$$

$$4 = -4r + 4v_2$$
$$16 = 4r + v_2$$

$$\longleftrightarrow \quad 20 = 5v^2$$
$$v_2 = 4;$$

$$16 = 4r + 4$$
$$4r = 12$$
$$r = 3$$

$$v_1 = 2(4) = 8$$
$$\vec{v} = (8, 4)$$
$$r = 3$$

(44) Prove: For any two vectors $\vec{v}$ and $\vec{w}$,

$$\|\vec{v}\|^2 \cdot \|\vec{w}\|^2 = (\vec{v}\cdot\vec{w})^2 + (\vec{v}\cdot\vec{w}_p)^2$$

Hint: Case I: $\vec{w} = \vec{0}$. Case 2: $\vec{w} = \vec{0}$; then

$$\vec{v} = \frac{(\vec{v}\cdot\vec{w})\vec{w} + (\vec{v}\cdot\vec{w}_p)\vec{w}_p}{\|\vec{w}\|^2}$$

Case 1: $\vec{w} = \vec{0} = (0,0)$, $\vec{w}_p = (0,0)$

$$\|\vec{v}\|^2 \cdot \|\vec{w}\|^2 = (v_1^2 + v_2^2) \cdot 0 = 0$$

$$(\vec{v}\cdot\vec{0})^2 + (\vec{v}\cdot\vec{0})^2 = 0$$

Case 2: $\vec{w} \neq \vec{0}$; $\vec{w} = (w_1, w_2)$, $\vec{w}_p = (-w_2, w_1)$

$$\vec{v} = \frac{[(v_1, v_2)\cdot(w_1, w_2)](w_1, w_2) + [(v_1, v_2)\cdot(-w_2, w_1)](-w_2, w_1)}{\|\vec{w}\|^2}$$

$$= \frac{(v_1w_1 + v_2w_2)(w_1, w_2) + (-v_1w_2 + v_2w_1)(-w_2, w_1)}{\|\vec{w}\|^2}$$

$v_2w_1^2$

$$= \frac{(w_1^2v_1 + w_1w_2v_2,\ v_1w_1w_2 + v_2w_2^2) + (v_1w_2^2 - w_1w_2v_2,\ -w_1w_2v_1 + v_2w_1^2)}{\|\vec{w}\|^2}$$

$$= \frac{(v_1(w_1^2 + w_2^2),\ v_2(w_1^2 + w_2^2))}{\|\vec{w}\|^2} = \frac{(v_1(\vec{w}\cdot\vec{w}),\ v_2(\vec{w}\cdot\vec{w}))}{\|\vec{w}\|^2}$$

$$\vec{V} = \frac{\vec{V}(\vec{W}\cdot\vec{W})}{\|\vec{W}\|^2}$$ This didn't get me anywhere.

I ought not expand $\vec{V}\cdot\vec{W}$ into $(v_1, v_2)\cdot(w_1, w_2)$ nor $\vec{V}\cdot\vec{W}_p$ into $(v_1, v_2)\cdot(-w_2, w_1)$.
Let's try leaving those in tact.

$$\vec{V} = \frac{(\vec{V}\cdot\vec{W})(w_1, w_2) + (\vec{V}\cdot\vec{W}_p)(-w_2, w_1)}{\|\vec{W}\|^2}$$

$$= \frac{[w_1(\vec{V}\cdot\vec{W}), w_2(\vec{V}\cdot\vec{W})] + [-w_2(\vec{V}\cdot\vec{W}_p), w_1(\vec{V}\cdot\vec{W}_p)]}{\|\vec{W}\|^2}$$

$$= \left(\frac{w_1(\vec{V}\cdot\vec{W}) - w_2(\vec{V}\cdot\vec{W}_p), w_1(\vec{V}\cdot\vec{W}_p) + w_2(\vec{V}\cdot\vec{W})}{\|\vec{W}\|^2}\right)$$

$$\|\vec{V}\|^2 = \vec{V}\cdot\vec{V} = \frac{[w_1(\vec{V}\cdot\vec{W}) - w_2(\vec{V}\cdot\vec{W}_p)]^2 + [w_1(\vec{V}\cdot\vec{W}_p) + w_2(\vec{V}\cdot\vec{W})]^2}{\|\vec{W}\|^4}$$

$$\|\vec{V}\|^2\cdot\|\vec{W}\|^4 = \left((\vec{V}\cdot\vec{W})^2\cdot w_1^2 + (\vec{V}\cdot\vec{W})^2 w_2^2 + (\vec{V}\cdot\vec{W}_p)^2 w_1^2 + (\vec{V}\cdot\vec{W}_p)^2\cdot w_2^2\right)$$

$$= (\vec{V}\cdot\vec{W})^2(w_1^2 + w_2^2) + (\vec{V}\cdot\vec{W}_p)^2(w_1^2 + w_2^2)$$

$\boxed{\text{Note that } (w_1^2 + w_2^2) = \vec{W}\cdot\vec{W} = \|\vec{W}\|^2}$

$$\|\vec{v}\|^2\|\vec{w}\|^4 = \|\vec{w}\|^2\left((\vec{v}\cdot\vec{w})^2 + (\vec{v}\cdot\vec{w_p})^2\right)$$

$$\|\vec{v}\|^2\|\vec{w}\|^2 = (\vec{v}\cdot\vec{w})^2 + (\vec{v}\cdot\vec{w_p})^2$$

---

(45) Show that $\|\vec{v}\|^2\|\vec{w}\|^2 \geq (\vec{v}\cdot\vec{w})^2$ follows from the assertion in exercise 44.

$(\vec{v}\cdot\vec{w_p})^2 \geq 0$ so $(\vec{v}\cdot\vec{w})^2 + (\vec{v}\cdot\vec{w_p})^2 \geq (\vec{v}\cdot\vec{w})^2$

Since $\|\vec{v}\|^2\|\vec{w}\|^2 = (\vec{v}\cdot\vec{w})^2 + (\vec{v}\cdot\vec{w_p})^2$

it follows that $\|\vec{v}\|^2\|\vec{w}\|^2 \geq (\vec{v}\cdot\vec{w})^2$

---

(46) Show that if $\vec{v}$ and $\vec{w}$ are parallel vectors, then $\|\vec{v}\|^2\|\vec{w}\|^2 = (\vec{v}\cdot\vec{w})^2$ follows from the assertion in exercise 44.

If $\vec{v}$ and $\vec{w}$ are parallel vectors, then $\vec{v}\cdot\vec{w_p} = 0$

so $\|\vec{v}\|^2\|\vec{w}\|^2 = (\vec{v}\cdot\vec{w})^2 + 0$

---

(47) Given $\boxed{\vec{v} \neq \vec{0} \text{ and } \vec{w} \neq \vec{0}}$. Prove: (a) if $\|\vec{v}\|\|\vec{w}\| = \vec{v}\cdot\vec{w}$, then $\vec{v}$ and $\vec{w}$ have the same direction.

(b) if $-\|\vec{v}\|\|\vec{w}\| = \vec{v}\cdot\vec{w}$, then $\vec{v}$ and $\vec{w}$ have opposite directions.

* This will require some writing space!

(9) If $\|\vec{v}\|\|\vec{w}\| = \vec{v}\cdot\vec{w}$, then $\vec{v}$ and $\vec{w}$ have the same direction.

Let $\vec{v} = (v_1, v_2)$ and $\vec{w} = (w_1, w_2)$

$$\|\vec{v}\|\|\vec{w}\| = \sqrt{(v_1^2+v_2^2)}\sqrt{(w_1^2+w_2^2)} = \sqrt{(v_1^2+v_2^2)(w_1^2+w_2^2)}$$

$$= \sqrt{v_1^2 w_1^2 + v_1^2 w_2^2 + v_2^2 w_1^2 + v_2^2 + w_2^2} = v_1 w_1 + v_2 w_2$$

Now we square both sides of the equation $\|\vec{v}\|\|\vec{w}\| = \vec{v}\cdot\vec{w}$:

$$v_1^2 w_1^2 + v_1^2 w_2^2 + v_2^2 w_1^2 + v_2^2 w_2^2 = v_1^2 w_1^2 + 2 v_1 v_2 w_1 w_2 + v_2^2 w_2^2$$

$$v_1^2 w_2^2 + v_2^2 w_1^2 - 2 v_1 v_2 w_1 w_2 = 0$$

Since $\vec{w} \neq \vec{0}$, w_1 or w_2 is not zero.

Suppose $w_2 \neq 0$: $v_2 = \frac{v_2}{w_2} w_2$ and $v_1 = \frac{v_2}{w_2} w_1$

$\therefore$ $\vec{v} = (r\cdot w_1, r\cdot w_2)$ where $r = \frac{v_2}{w_2}$ and the vectors are parallel. They have the same direction if $r > 0$:

Since $\vec{v}\cdot\vec{w} = \|\vec{v}\|\|\vec{w}\| > 0$, $v_1 w_1 + v_2 w_2 > 0$

$$\left(\frac{v_2}{w_2} w_1\right) w_1 + \left(\frac{v_2}{w_2} w_2\right) w_2 = r\cdot w_1 w_1 + r w_2 w_2 > 0$$

$$r(w_1^2 + w_2^2) > 0, \quad r\|\vec{w}\|^2 > 0 \quad \therefore r > 0$$

(b) If $-\|\vec{v}\|\|\vec{w}\| = \vec{v}\cdot\vec{w}$, then $\vec{v}$ and $\vec{w}$ have opposite directions.

$$\|\vec{v}\|\|\vec{w}\| = -\vec{v}\cdot\vec{w} = -(v_1, v_2)\cdot(w_1, w_2) = -(v_1w_1 + v_2w_2)$$

$$\|\vec{v}\|\|\vec{w}\| = -v_1w_1 - v_2w_2 \longleftrightarrow (\|\vec{v}\|\|\vec{w}\|)^2 = (-v_1w_1 - v_2w_2)^2$$

From part (a): $(\|\vec{v}\|\|\vec{w}\|)^2 = v_1^2w_1^2 + v_1^2w_2^2 + v_2^2w_1^2 + v_2^2w_2^2$

$$= (-v_1w_1 - v_2w_2)^2 = v_1^2w_1^2 + 2v_1v_2w_1w_2 + v_2^2w_2^2$$

$$v_1^2w_2^2 + v_2^2w_1^2 - 2v_1v_2w_1w_2 = 0$$

Since $\vec{W} \neq 0$, w_1 or w_2 is not zero. Suppose $w_2 \neq 0$:

$v_2 = \frac{v_2}{w_2}w_2$ and $v_1 = \frac{v_2}{w_2}w_1$

$\therefore \vec{v} = (r\cdot w_1, rw_2)$ where $r = \frac{v_2}{w_2}$ and the vectors are parallel. Up to this point, the proof has been the same as in part (a). Now, $\vec{v}$ and $\vec{w}$ have opposite directions if $r < 0$.

Since $-\vec{v}\cdot\vec{w} = \|\vec{v}\|\|\vec{w}\| < 0$, $v_1w_1 + v_2w_2 < 0$

$$\left(\frac{v_2}{w_2}w_1\right)w_1 + \left(\frac{v_2}{w_2}w_2\right)w_2 = r\cdot w_1^2 + r\cdot w_2^2 < 0$$

$$r(w_1^2 + w_2^2) < 0 \longleftrightarrow r\cdot\|\vec{w}\|^2 < 0$$

$\therefore$ $r < 0$ since $\|\vec{w}\|^2 > 0$

(48) On the basis of the assertion in Exercise 47 prove that for $\vec{v} \neq \vec{0}$ and $\vec{w} \neq \vec{0}$,

$\|\vec{v}\| + \|\vec{w}\| = \|\vec{v} + \vec{w}\|$ if and only if $\vec{v}$ and $\vec{w}$ have the same direction.

$\vec{v}$ and $\vec{w}$ are parallel if $\vec{v} = r \cdot \vec{w}$, and vectors $\vec{v}$ and $\vec{w}$ have the same direction if $r > 0$.

We begin with $\|\vec{v} + \vec{w}\|$ and $\boxed{\vec{v} = r \cdot \vec{w}}$

$$\|\vec{v}+\vec{w}\| = \|r \cdot \vec{w} + \vec{w}\| = \|(r+1)\vec{w}\| = (r+1)\|\vec{w}\|$$

$$= r\|\vec{w}\| + \|\vec{w}\| = \|r \cdot \vec{w}\| + \|\vec{w}\|$$

$$= \|\vec{v}\| + \|\vec{w}\|,$$

If $\|\vec{v}+\vec{w}\| = \|\vec{v}\| + \|w\|$,

$$\|(v_1 + w_1, v_2 + w_2)\| = \|\vec{v}\| + \|\vec{w}\|$$

$$\sqrt{(v_1+w_1)^2 + (v_2+w_2)^2} = \sqrt{v_1^2+v_2^2} + \sqrt{w_1^2+w_2^2}$$

$$(v_1+w_1)^2 + (v_2+w_2)^2 = v_1^2 + v_2^2 + 2\left(\sqrt{v_1^2+v_2^2}\right)\left(\sqrt{w_1^2+w_2^2}\right)$$

$$+ w_1^2 + w_2^2$$

$(v_1+w_1)^2 + (v_2+w_2)^2 = v_1^2 + 2v_1w_1 + w_1^2 + v_2^2 + 2v_2w_2 + w_2^2$

$2\sqrt{v_1^2+v_2^2}\sqrt{w_1^2+w_2^2} = 2(v_1w_1 + v_2w_2) = 2(\vec{v}\cdot\vec{w})$

$\vec{v}\cdot\vec{w} = \sqrt{(v_1^2+v_2^2)}\cdot\sqrt{(w_1^2+w_2^2)} = \|\vec{v}\|\cdot\|\vec{w}\|$

Since $\|\vec{v}\|\|\vec{w}\| = \vec{v}\cdot\vec{w}$, $\vec{v}$ and $\vec{w}$ have the same direction.

Chapter 4 Test

4-1 ① Find x and y if $(3x-2, y+3) = (7, -2)$

$7 = 3x-2$ and $-2 = y+3$

$3x = 9 \leftrightarrow x = 3$ and $y = -5$

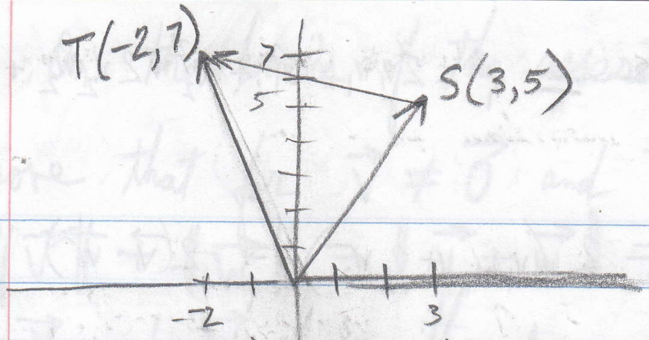
② If $A = \{3, 2\}$ and $B = \{-1, 4\}$, specify by roster $A\times B$.

$A\times B = \{(3,-1), (3,4), (2,-1), (2,4)\}$

4-2 ③ Name the vector represented by $\overrightarrow{MN}$ when the respective coordinates of M and N are $(3,-2)$ and $(5,6)$.

$\overrightarrow{MN} = (5,6) - (3,-2) = (5,6) + (-3,2) = (2,8)$

④ Sketch the vectors $(3,5)$ and $(-2,7)$ in standard position and label the terminal points S and T respectively. Name the vector represented by $\overrightarrow{ST}$.



$\overrightarrow{ST} = (-2,7) - (3,5) = (-2,7) + (-3,-5)$
$= (-5,2)$

4-3 (5) Find r and s in the vector sum:
$(r,s) + (-3,2) = (4,-1)$

$(r-3, s+2) = (4,-1)$
$r-3 = 4 \longleftrightarrow r = 7$
$s+2 = -1 \longleftrightarrow s = -3$

(6) If $\vec{v} = (3,-2)$ and $\vec{t} = (5,-4)$, determine the vector $\vec{v} - \vec{t}$.
$\vec{v} - \vec{t} = (3,-2) - (5,-4) = (3,-2) + (-5,4)$
$= (-2,2)$

4-4 (7) Find the norm of the vector $(3,-2)$.

$\|(3,-2)\| = \sqrt{3^2 + (-2)^2} = \sqrt{13}$

(8) If $\vec{r} = (3,5)$ and $\vec{s} = (-2,1)$, find the scalar $\|\vec{r} + \vec{s}\|$.
$\|(3,5) + (-2,1)\| = \|(1,6)\| = \sqrt{1^2 + 6^2} = \sqrt{37}$

4-5 9 Are $(5,-3)$ and $(-10,6)$ vectors having the same direction, opposite direction, or neither.

$(-10,6) = r(5,-3)$, $-10 = 5r$, $r = -2$
$6 = -3r$, $r = -2 < 0$ $\therefore$ opposite directions

10 Find the unit vector with the same direction as $(3,-4)$.

$$\frac{(3,-4)}{\sqrt{3^2+(-4)^2}} = \frac{(3,-4)}{5} = \left(\frac{3}{5}, \frac{-4}{5}\right)$$

4-6 11 Find the inner product of $(3,-2)$ and $(5,3)$.

$(3,-2)\cdot(5,3) = 3\cdot5 + (-2)(3) = 15 - 6 = 9$

12 For what value of k will $(3,-4)$ and $(k,6)$ be parallel vectors?

$(k,6)\cdot(4,3) = 0 \leftrightarrow 4k + 18 = 0 \leftrightarrow 4k = -18$
$\leftrightarrow k = -\frac{9}{2}$

or $(3,-4)\cdot(-6,k) = 0 \leftrightarrow -18 - 4k = 0 \leftrightarrow -4k = 18 \rightarrow k = -\frac{9}{2}$

4-7 13 If $\vec{v} = (2,-3)$, what is $\vec{v}_p$?
$\vec{v}_p = (3,2)$

14 What must be true if (a,b) and (c,d) are nonzero perpendicular vectors? $a\cdot c + bd = 0$

4-8 15 Given $\vec{v} = (6,-2)$. Express $\vec{v}$ as a linear combination of $\vec{u} = (1,0)$ and $\vec{u}_p = (0,1)$.

$$\vec{v} = 6(1,0) - 2(0,1) = 6\vec{u} - 2\vec{u}_p$$

16 Given $\vec{s} = (4,5)$. Determine the vector components of $\vec{s}$ parallel to and perpendicular to $\vec{w}$ if $\vec{w} = (-2,3)$.

$$\overline{\text{Comp}}_{w}\,\vec{s} = \frac{\vec{w}\cdot\vec{s}}{\vec{w}\cdot\vec{w}}\vec{w} = \frac{(-2,3)\cdot(4,5)}{(-2,3)\cdot(-2,3)}(-2,3)$$

$$= \frac{-8+15}{4+9}(-2,3) = \frac{7}{13}(-2,3) = \left(-\frac{14}{13}, \frac{21}{13}\right)$$

$$\vec{w}_p = (-3,-2)$$

$$\overline{\text{Comp}}_{w_p}\,\vec{s} = \frac{\vec{w}_p\cdot\vec{s}}{\vec{w}\cdot\vec{w}}\vec{w}_p = \frac{(-3,-2)\cdot(4,5)}{13}(-3,-2)$$

$$= \frac{-12-10}{13}(-3,-2) = \frac{-22}{13}(-3,-2) = \left(\frac{66}{13}, \frac{44}{13}\right)$$

$$\overline{\text{Comp}}_{w}\,\vec{s} + \overline{\text{Comp}}_{w_p}\,\vec{s} = \left(-\frac{14}{13}, \frac{21}{13}\right) + \left(\frac{66}{13}, \frac{44}{13}\right)$$

$$= \left(\frac{52}{13}, \frac{65}{13}\right) = (4,5)$$

5 PLANE ANALYTIC GEOMETRY OF POINTS AND LINES

VECTORS, POINTS, AND LINES

5-1 POINTS IN THE PLANE

For the given points P and T, compute distance (P, T).

① $P(0,0)$; $T(1,3)$: $dist(P,T) = \sqrt{(1-0)^2+(3-0)^2}$
$= \sqrt{1^2+3^2} = \sqrt{10}$

② $P(0,-3)$; $T(2,0)$: $dist(P,T) = \sqrt{(2-0)^2+(0-(-3))^2}$
$= \sqrt{2^2+3^2} = \sqrt{13}$

③ $P(2,3)$; $T(-2,1)$: $dist(P,T) = \sqrt{(-2-2)^2+(1-3)^2}$
$= \sqrt{(-4)^2+(-2)^2} = \sqrt{16+4} = \sqrt{20} = 2\sqrt{5}$

④ $P(-3,5)$; $T(4,3)$: $dist(P,T) = \sqrt{(4-(-3))^2+(3-5)^2}$

$= \sqrt{7^2+(-2)^2} = \sqrt{53}$

⑤ $P(3\sqrt{2}, \sqrt{3})$; $T(\sqrt{2}, -3\sqrt{3})$
$dist(P,T) = \sqrt{(\sqrt{2}-3\sqrt{2})^2+(-3\sqrt{3}-\sqrt{3})^2}$
$= \sqrt{(-2\sqrt{2})^2+(-4\sqrt{3})^2} = \sqrt{8+48} = \sqrt{56}$
$= 2\sqrt{14}$

⑥ $P(a,b)$; $T(c,d)$: $dist(P,T) = \sqrt{(c-a)^2+(d-b)^2}$
$= \sqrt{c^2-2ac+a^2+d^2-2bd+d^2}$

(7) $P(a,b)$; $T(2a,3b)$

$dist(P,T) = \sqrt{(2a-a)^2+(3b-b)^2} = \sqrt{a^2+4b^2}$

(8) $P(a-b,\ c+d)$; $T(a+b,\ c-d)$

$dist(P,T) = \sqrt{(a+b-a+b)^2+(c-d-c-d)^2}$

$= \sqrt{(2b)^2+(-2d)^2} = \sqrt{4b^2+4d^2}$

$= 2\sqrt{b^2+d^2}$

For the given points A, B, and C determine whether < or = is the appropriate symbol in the assertion $dist(A,B) \leq dist(A,C) + dist(C,B)$

(9) $A(0,0)$; $B(-6,8)$; $C(0,8)$

$distance(A,B) = \sqrt{(-6)^2+(8)^2} = \sqrt{36+64} = 10$

$distance(A,C) = \sqrt{(0)^2+8^2} = 8$

$distance(C,B) = \sqrt{(-6)^2+(0^2)} = 6$

$\therefore\ 10 < 8+6 = 14$

(10) $A(0,0)$; $B(6,2)$; $C(2,6)$

$dist(A,B) = \sqrt{6^2+2^2} = \sqrt{40} = 2\sqrt{10}$

$dist(A,C) = \sqrt{2^2+6^2} = 2\sqrt{10}$

$dist(C,B) = \sqrt{(6-2)^2+(2-6)^2} = \sqrt{4^2+(-4)^2} = \sqrt{32}$

$= 4\sqrt{2}$

$\therefore\ 2\sqrt{10} < 2\sqrt{10} + 4\sqrt{2}$

(11) $A(8,6)$; $B(8,-4)$; $C(10,-4)$

$dist(A,B) = \sqrt{(8-8)^2+(-4-6)^2} = 10$

$dist(A,C) = \sqrt{(10-8)^2+(-4-6)^2} = \sqrt{104} = 2\sqrt{26}$

$dist(C,B) = \sqrt{(8-10)^2+(-4-(-4))^2}$

$= \sqrt{(-2)^2+0^2} = 2$

$\therefore\ 10 < 2\sqrt{26} + 2 \approx 14.198$

(12) $A(2,0)$; $B(5,0)$; $C(-1,4)$

$dist(A,B) = \sqrt{(5-2)^2+0^2} = 3$

$dist(A,C) = \sqrt{(-1-2)^2+4^2} = 5$

$dist(C,B) = \sqrt{(5+1)^2+(0-4)^2} = \sqrt{36+16} = \sqrt{52} = 2\sqrt{13}$

$\therefore\ 3 < 5 + 2\sqrt{3}$

(13) $A(-4,3)$; $B(2,-5)$; $C(3,2)$

$$dist(A,B) \le dist(A,C) + dist(C,B)$$

$$dist(A,B) = \sqrt{(2+4)^2 + (-5-3)^2} = \sqrt{36+64} = 10$$
$$dist(A,C) = \sqrt{(3+4)^2 + (2-3)^2} = \sqrt{50} = 5\sqrt{2}$$
$$dist(C,B) = \sqrt{(2-3)^2 + (-5-2)^2} = \sqrt{50} = 5\sqrt{2}$$

$$\therefore\ 10 < 5\sqrt{2} + 5\sqrt{2} = 10\sqrt{2}$$

(14) $A(5,3)$; $B(-1,-5)$; $C(2,-1)$

$$dist(A,B) = \sqrt{(-1-5)^2 + (-5-3)^2} = \sqrt{(-6)^2 + (-8)^2}$$
$$= \sqrt{36+64} = 10$$
$$dist(A,C) = \sqrt{(2-5)^2 + (-1-3)^2} = \sqrt{(-3)^2 + (-4)^2}$$
$$= \sqrt{9+16} = 5$$

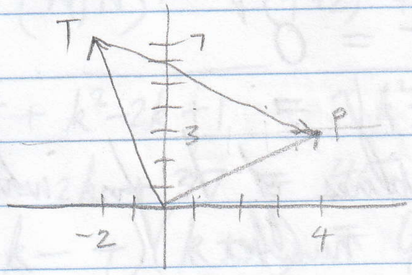
$$dist(C,B) = \sqrt{(-1-2)^2 + (-5-2)^2} = \sqrt{9+49}$$
$$= \sqrt{58}$$

$$\therefore\ 10 < 5 + \sqrt{58} \approx 12.61577$$

(15) Show that the distance between $P(a,b)$ and $T(a,c)$ is not dependent upon a.

$$dist(P,T) = \sqrt{(a-a)^2+(c-b)^2} = \sqrt{c^2-2bc+b^2}$$

(16) Represent the ordered pairs $P(4,3)$ and $T(-2,7)$ as arrows in standard position. Find the length of the arrow between P and T.



$$dist(P,T) = \sqrt{(-2-4)^2+(7-3)^2}$$
$$= \sqrt{(-6)^2+4^2} = \sqrt{36+16}$$
$$= \sqrt{52} = 2\sqrt{13}$$

(17) Write an equation which expresses the fact that $P(x,y)$ is equidistant from $R(-2,6)$ and $S(4,8)$.

$$dist(R,P) = dist(S,P)$$

$$\sqrt{(x+2)^2+(y-6)^2} = \sqrt{(x-4)^2+(y-8)^2}$$
$$x^2+4x+4+y^2-12y+36 = x^2-8x+16+y^2-16y+64$$
$$4y = -12x+40 \leftrightarrow y = -3x+10$$

(18) Write an equation which expresses the fact that the distance from $P(x,y)$ to $R(2,3)$ is twice the distance from $P(x,y)$ to $S(-6,4)$.

$$dist(P,R) = 2\cdot dist(P,S)$$
$$\sqrt{(2-x)^2+(3-y)^2} = 2\cdot\sqrt{(-6-x)^2+(4-y)^2}$$

$$\sqrt{(2-x)^2+(3-y)^2} = 2\cdot\sqrt{(-6-x)^2+(4-y)^2}$$

$$\sqrt{(x-2)^2+(y-3)^2} = 2\cdot\sqrt{(x+6)^2+(y-4)^2}$$

$$(x-2)^2+(y-3)^2 = 4\left[(x+6)^2+(y-4)^2\right]$$

$$x^2-4x+4+y^2-6y+9 = 4(x^2+12x+36+y^2-8y+16)$$
$$x^2-4x+4+y^2-6y+9 = 4x^2+48x+144+4y^2-32y+64$$
$$3x^2+52x+3y^2-26y+195 = 0$$

In each exercise, find the values of k for which $dist(M,N)$ is the given number.

(19) $M(6,k)$; $N(2,0)$; $dist(M,N)=5$

$$dist(M,N) = \sqrt{(2-6)^2+(0-k)^2} = 5$$

$$\sqrt{4^2+(-k)^2} = \sqrt{16+k^2} = 5$$
$$16+k^2 = 25 \longleftrightarrow k^2 = 9$$
$$k = \pm 3$$

(20) $M(2,k)$; $N(4,2k)$; $dist(M,N)=\sqrt{13}$

$$dist(M,N) = \sqrt{(4-2)^2+(2k-k)^2} = \sqrt{13}$$

$$\sqrt{2^2+k^2} = \sqrt{13} \longleftrightarrow 4+k^2 = 13 \longleftrightarrow k^2 = 9$$
$$k = \pm 3$$

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(21) $M(3,2)$; $N(4,k)$; $dist(M,N) = \sqrt{17}$

$dist(M,N) = \sqrt{(4-3)^2 + (k-2)^2} = \sqrt{17}$

$\sqrt{1 + k^2 - 4k + 4} = \sqrt{k^2 - 4k + 5} = \sqrt{17}$

$k^2 - 4k + 5 = 17 \longleftrightarrow k^2 - 4k - 12 = 0$

$(x-6)(x+2) = 0 \quad \therefore k = 6$ or $k = -2$

(22) $M(-2,1)$; $N(4,k)$; $dist(M,N) = 3\sqrt{5}$

$dist(M,N) = \sqrt{(4+2)^2 + (k-1)^2} = 3\sqrt{5}$

$\sqrt{6^2 + k^2 - 2k + 1} = \sqrt{k^2 - 2k + 37} = 3\sqrt{5}$

$k^2 - 2k + 37 = 45 \longleftrightarrow k^2 - 2k - 8 = 0$

$(k-4)(k+2) = 0 \longleftrightarrow k = 4$ or $k = -2$

(23) $M(5,k)$, $N(2,5)$; $dist(M,N) = \sqrt{34}$

$dist(M,N) = \sqrt{(2-5)^2 + (5-k)^2} = \sqrt{34}$

$3^2 + (k-5)^2 = 34 \longleftrightarrow k^2 - 10k + 25 = 25$

$k^2 - 10k = 0 \longleftrightarrow k(k-10) = 0 \longleftrightarrow k = 0$ or $k = 10$

(24) $M(2a,k)$; $N(a,3b)$; $dist(M,N) = \sqrt{a^2+b^2}$

$dist(M,N) = \sqrt{(a-2a)^2 + (3b-k)^2} = \sqrt{a^2+b^2}$

$a^2 + 9b^2 - 6bk + k^2 = a^2 + b^2$

$8b^2 - 6bk + k^2 = 0 \longleftrightarrow k^2 - 6bk + 8b^2 = 0$

$k^2 - 6bk + (-3b)^2 = -8b^2 + 9b^2 = b^2$

$(k-3b)^2 = b^2 \longleftrightarrow k - 3b = \pm b \longleftrightarrow k = 4b$ or $k = 2b$

(25) Given $R(2,2)$ and $S(6,6)$.
Find the coordinates of a point K whose representation in the coordinate plane lies on the x-axis and is such that
$dist(R,K) = dist(K,S)$

Solution: Since K is on the x-axis, its ordinate y must be 0.

$dist(R,K) = dist(K,S)$

$\sqrt{(x-2)^2+(0-2)^2} = \sqrt{(6-x)^2+(6-0)^2}$

$\sqrt{x^2-4x+4+4} = \sqrt{36-12x+x^2+6^2}$

$x^2-4x+8 = x^2-12x+72$

$8x = 64 \longleftrightarrow x = 8$

$K = (8, 0)$

(26) Given $A(-2,8)$ and $B(4,2)$.
Find the coordinates of point P whose representation in the coordinate plane lies on the y-axis and such that
$dist(A,P) = dist(P,B)$
$x=0$ so

$\sqrt{(0+2)^2+(y-8)^2} = \sqrt{(4-0)^2+(2-y)^2}$

$2^2+y^2-16y+64 = 16+4-4y+y^2$

$48 = 12y \longleftrightarrow y = 4$

$P=(0,4)$

(27) Show that $P\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ is equidistant from $M(x_1, y_1)$ and $N(x_2, y_2)$.

$$dist(P, M) = dist(N, P)$$

$$\sqrt{\left(x_1-\frac{x_1+x_2}{2}\right)^2+\left(y_1-\frac{y_1+y_2}{2}\right)^2} = \sqrt{\left(\frac{x_1+x_2}{2}-x_2\right)^2+\left(\frac{y_1+y_2}{2}-y_2\right)^2}$$

$$\left(\frac{2x_1-x_1-x_2}{2}\right)^2+\left(\frac{2y_1-y_1-y_2}{2}\right)^2 = \left(\frac{x_1+x_2-2x_2}{2}\right)^2+\left(\frac{y_1+y_2-2y_2}{2}\right)^2$$

$$\left(\frac{x_1-x_2}{2}\right)^2+\left(\frac{y_1-y_2}{2}\right)^2 = \left(\frac{x_1-x_2}{2}\right)^2+\left(\frac{y_1-y_2}{2}\right)^2$$

$$\therefore\ dist(P, M) = dist(N, P)$$

(28) Prove that, if P, S, and T are vectors, then $dist(P, S) = dist(P+kT, S+kT)$ for every $k \in \mathbb{R}$.

$P = (x_1, y_1)$, $S = (x_2, y_2)$, $T = (x_3, y_3)$

$$dist(P, S) = \|S-P\|$$

$$dist(P+kT, S+kT) = \|(S+kT)-(P+kT)\|$$

$$= \|S-P+(k-k)T\| = \|S-P\|$$

$$\therefore\ dist(P, S) = dist(P+kT, S+kT)$$

(29) Prove that, if P, S, and T are vectors, then

$$\text{dist}(P,S) = d(P+T, S+T)$$

$$P = (x_1, y_1),\ S = (x_2, y_2),\ T = (x_3, y_3)$$

$$\text{dist}(P,S) = \|S-P\|$$

$$\text{dist}(P+T, S+T) = \|(S+T)-(P+T)\| = \|S-P+(T-T)\|$$

$$= \|S-P\| \quad \therefore \text{dist}(P,S) = \text{dist}(P+T, S+T).$$

(30) Prove that if P, S, and T are vectors, then

$$\text{dist}(P,S) \stackrel{?}{=} \text{dist}(P-T, S-T)$$

$$\text{dist}(P,S) = \|S-P\|$$

$$\text{dist}(P-T, S-T) = \|(S-T)-(P-T)\|$$

$$= \|S-P+(T-T)\| = \|S-P\|$$

$$\therefore \text{dist}(P,S) = \text{dist}(P-T, S-T)$$

$\boxed{5-2}$ LINES IN THE PLANE

① Is a line a set of points? yes

a set of vectors? yes

a set of ordered pairs? yes

A set $\mathcal{L}$ of points $X(x,y)$ in $\mathbb{R}\times\mathbb{R}$ is called a line if there exists a point P and a nonzero vector $\vec{v}$ such that

$$\mathcal{L} = \{X : X = P + t\vec{v},\ t \in \mathbb{R}\}$$

(2) Can two different pairs of parametric equations identify the same line?

Yes; since each line contains many points, any one of these may be used as P in the equation $X = P + t\vec{v}$

Determine a vector equation of the line $\mathcal{L}$ through P with direction vector $\vec{v}$

(3) $P(3,4)$; $\vec{v} = (1,3)$

$\mathcal{L} = \{ P + t\vec{v} \} = \{ (3,4) + t(1,3) \}$

$= \{ (3+t, 4+3t) \}$

$(x,y) = (3+t, 4+3t)$

(4) $P(0,-2)$; $\vec{v} = (2,5)$

$(x,y) = (0,-2) + t(2,5) = (2t, -2+5t)$

(5) $P(-5,4)$; $\vec{v} = (-5,3)$

$(x,y) = (-5,4) + t(-5,3) = (-5-5t, 4+3t)$

(6) $P(-1,-3)$; $\vec{v} = (-1,2)$

$(x,y) = (-1,-3) + t(-1,2) = (-1-t, -3+2t)$

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(1) Since there is a one-to-one correspondence between $\mathbb{R} \times \mathbb{R}$ and all points of a plane, and vectors are members of $\mathbb{R} \times \mathbb{R}$, vectors and points can be treated as identical. Technically one could say that a point is different from an ordered pair of members assigned to it.

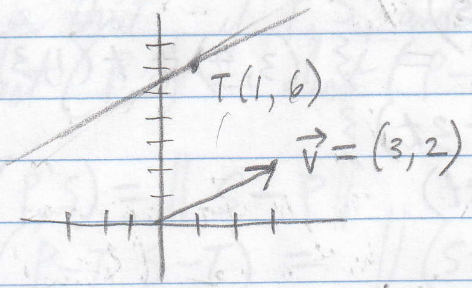
(7) $P(-2,1)$; $\vec{v} = (-2,3)$

$$(x,y) = (-2,1) + t(-2,3) = (-2-2t, 1+3t)$$

(8) $P(3,-4)$; $\vec{v} = (-3,-1)$

$$(x,y) = (3,-4) + t(-3,-1) = (3-3t, -4-t)$$

Assume that in the given coordinate plane representation $\vec{v}$ is a direction vector of $\mathcal{L}$.



(9) Write a vector equation of $\mathcal{L}$.

$$\mathcal{L} = \{ T + t\vec{v} \} = \{ (1,6) + t(3,2) \}$$
$$= \{ (1+3t, 6+2t) \}$$

or $(x,y) = (1+3t, 6+2t)$

(10) Is $(-2,4) \in \mathcal{L}$? yes, when $t = -1$

$$(x,y) = (1+3(-1), 6+2(-1)) = (-2,4)$$

(11) For what value of y is $(4,y) \in \mathcal{L}$?

$x = 4 = 1+3t$ $\quad \therefore y = 6 + 2(1) = 8$

$3t = 3 \leftrightarrow t = 1$

(12) For what value of x is $(x, 0) \in \mathcal{L}$?

$y = 0 = 6 + 2t$ $\therefore$ $x = 1 + 3(-3) = -8$

$2t = -6 \leftrightarrow t = -3$

Notes: Another way to approach exercise 10.

Is $(-2, 4) \in \mathcal{L}$?

Yes, $X - T = (-2, 4) - (1, 6) = (-2-1, 4-6) = (-3, -2)$

Since $(-3, -2) = -(3, 2)$, $(-3, -2) = t(3, 2)$

with $t = -1$. I started with this conclusion, but it is more methodical to think "X − P".

X lies on $\mathcal{L}$ if and only if $X - P$ is parallel to $\vec{v}$

Also, with exercise 11, I used the vector equation of $\mathcal{L}$ and the given value of x. With more formal notation:

Let $(4, y) = X$

$X \in \mathcal{L}$ means that $(X - P) \cdot \vec{v}_p = 0$.

$$\boxed{\begin{array}{l} T = (1, 6) \\ \vec{v} = (3, 2) \end{array}}$$

$X - T = (4, y) - (1, 6) = (3, y - 6)$

and $\vec{v}_p = (-2, 3)$

$(X - T) \cdot \vec{v}_p = (3, y-6) \cdot (-2, 3) = 0$

$(3)(-2) + (y-6)(3) = -6 + 3y - 18 = 3y - 24 = 0$

$3y = 24 \leftrightarrow y = 8$

Although my initial method provided the solution more quickly, the purpose of the exercise is to use $(X - P) \cdot \vec{v}_p = 0$ and then solve for y

Likewise with exercise 12, to find the value of x for which $(x, 0) \in \mathcal{L}$.

$$\boxed{\begin{array}{l} T = (1,6) \\ \vec{v} = (3,2) \end{array}}$$

Let $(x, 0) = X$

$X \in \mathcal{L}$ means that $(X - T) \cdot \vec{v}_p = 0$

$$[(x, 0) - (1, 6)] \cdot (-2, 3) = 0$$

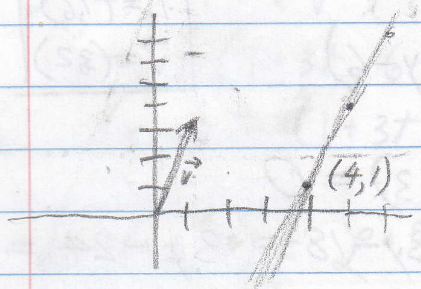
$$(x-1, -6) \cdot (-2, 3) = 0$$

$$(x-1)(-2) + (-6)(3) = -2x + 2 - 18 = 0$$

$$2x = -16 \longleftrightarrow x = -8$$

Represent the given line $\mathcal{L}$ in the coordinate plane and name three points $P(a, b)$ such that $P(a, b) \in \mathcal{L}$.

(13) $\mathcal{L} = \{ (4,1) + t(1,3) \}$



t	X
0	(4,1)
1	(5,4)
2	(6,7)

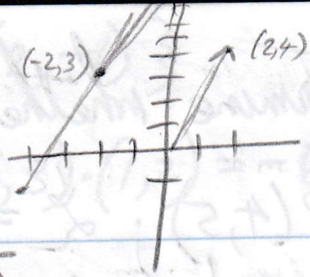
$$(x, y) = (4 + t, 1 + 3t)$$

$$x = 4 + t$$

$$y = 1 + 3t$$

(14) $\mathcal{L} = \{(-2,3) + t(2,4)\}$

$(x,y) = (-2+2t, 3+4t)$



$x = -2 + 2t$

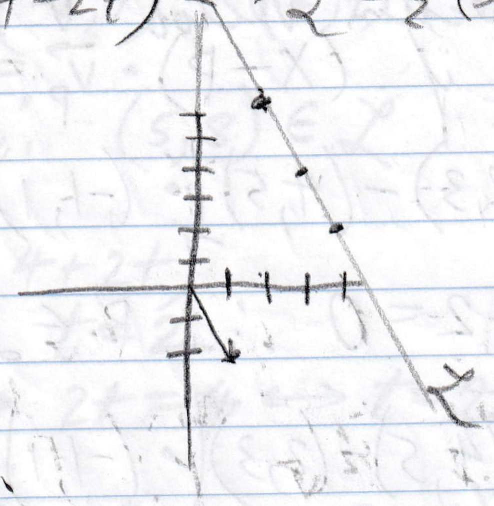
$y = 3 + 4t$

t	X
0	$(-2,3)$
1	$(0,7)$
-1	$(-4,-1)$

(15) $(x,y) = (3+t, 4-2t) \longleftrightarrow \mathcal{L} = \{(3,4) + t(1,-2)\}$

$x = 3 + t$

$y = 4 - 2t$

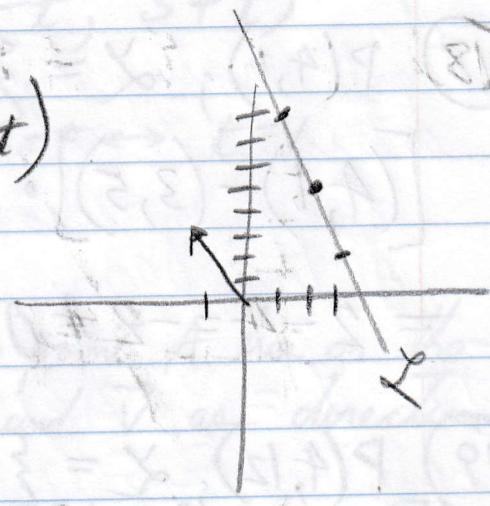


t	X
0	$(3,4)$
1	$(4,2)$
-1	$(2,6)$

(16) $(x,y) = (2-t, 5+3t)$

$x = 2 - t$

$y = 5 + 3t$



$\mathcal{L} = \{(2,5) + t(-1,3)\}$

t	X
0	$(2,5)$
1	$(1,8)$
-1	$(3,2)$

Determine whether the point P lies on the line $\mathcal{L}$.

(17) $P(4,5)$; $\mathcal{L} = \{(2,3) + t(1,1)\}$

Here $\mathcal{L} = \{X : X + t\vec{v}\}$ and $P = (4,5)$.

A point X lies on the line through P with direction vector $\vec{v} = (v_1, v_2)$ if and only if $(X-P)\cdot\vec{v}_p = 0$, where $\vec{v}_p = (-v_2, v_1)$.

$[(2,3)-(4,5)]\cdot(-1,1) = (-2,-2)\cdot(-1,1)$

$= 2-2 = 0 \therefore P \in \mathcal{L}$

or $[(4,5)-(2,3)]\cdot(-1,1) = (2,2)\cdot(-1,1) = 0$

(18) $P(4,1)$; $\mathcal{L} = \{(3,5) + t(1,-2)\}$

$[(4,1)-(3,5)]\cdot(2,1) = (1,-4)\cdot(2,1)$

$= 2-4 = -2 \neq 0 \therefore$ no, $(4,1) \notin \mathcal{L}$

(19) $P(4,12)$, $\mathcal{L} = \{(-2,3) + t(2,3)\}$

$[(4,12)-(-2,3)]\cdot(-3,2) = (6,9)\cdot(-3,2)$

$= -18+18 = 0 \therefore (4,12) \in \mathcal{L}$

(20) $P(-1,4)$; $\mathcal{L} = \{(5,1) + t(2,-1)\}$

$[(-1,4)-(5,1)]\cdot(1,2) = (-6,3)\cdot(1,2) = -6+6 = 0$

$\therefore\ (-1,4) \in \mathcal{L}$

(21) $P(5,8)$; $(x,y) = (3+t,\ 4+2t)$

$\mathcal{L} = \{(3,4) + t(1,2)\}$

$[(5,8)-(3,4)]\cdot(-2,1) = (2,4)\cdot(-2,1)$

$= -4+4 = 0 \quad \therefore\ (5,8) \in \mathcal{L}$

Another method:

$(5,8) = (3+t,\ 4+2t)$

$5 = 3+t \leftrightarrow t = 2$

$8 = 4+2t \leftrightarrow 2t = 4 \leftrightarrow t = 2$

(22) $P(5,4)$, $(x,y) = (4-t,\ 3+t)$

$(5,4) = (4-t,\ 3+t)$

$5 = 4-t \leftrightarrow t = -1$, $4 = 3+t \leftrightarrow t = 1$

$\therefore$ No, $(5,4) \notin \mathcal{L}$

2{23,..26} determine whether the point A lies on the ray having P as endpoint and $\vec{v}$ as direction vector.

(23) $A(2,6)$, $P(0,4)$, $\vec{v} = (1,1)$

$\mathcal{S} = \{X : X = (0,4) + t(1,1),\ t \geq 0\}$

is the vector equation. $A - P = (2,6) - (0,4) = (2,2)$

$= 2(1,1) = 2\vec{v}$, $2 > 0 \therefore A \in \mathcal{S}$

(24) $A(7,-3)$; $P(5,0)$; $\vec{v} = (-2,3)$

$A - P = (7-5, -3-0) = (2,-3) = -1(-2,3)$

$= -1\vec{v}$ so: $-1 < 0$ ∴ no, $A \notin \ell$

(25) $A(2,8)$; $P(-1,4)$; $\vec{v} = (3,4)$

$A - P = (2+1, 8-4) = (3,4) = 1\vec{v}$

so $1 > 0$ ∴ yes, $A \in \ell$

(26) $A(-5,1)$; $P(1,5)$; $\vec{v} = (-2,-1)$

$A - P = (-5-1, 1-5) = (-6,-4)$

$= 3\left(-2, -\frac{4}{3}\right) \neq t\vec{v}$ for any $t \in R$

Without resorting to a coordinate plane representation, determine whether the pair of equations given in each exercise specify the same line.

(27) $\mathcal{L} = \{(2,3) + t(3,-2)\}$
and $\mathcal{L} = \{(2,3) + t(6,-4)\}$

4/14 Yes. Both lines contain $(2,3)$.
Since $2(3,-2) = (6,-4)$, the lines have parallel direction vectors.

(28) $\mathcal{L} = \{(4,-6) + t(3,1)\}$ and $\mathcal{J} = \{(7,-5) + t(-3,-1)\}$

$(4,-6) + t(3,1) = (4+3t, -6+t)$
$(7,-5) + t(-3,-1) = (7-3t, -5-t)$

The two lines are equal if $4+3t = 7-3t$ and $-6+t = -5-t$.

$4+3t = 7-3t \longleftrightarrow 6t = 3 \longleftrightarrow t = \frac{1}{2}$
$-6+t = -5-t \longleftrightarrow 2t = 1 \longleftrightarrow t = \frac{1}{2}$
The lines are equal for $t = \frac{1}{2}$

(29) $(x,y) = (4+3r, -6+r)$ and $(x,y) = (7-3r, -5-r)$
These two lines are equal if $4+3r = 7-3r$ and $-6+r = -5-r$.

$4+3r = 7-3r \longleftrightarrow 6r = 3 \longleftrightarrow r = \frac{1}{2}$
$-6+r = -5-r \longleftrightarrow 2r = 1 \longleftrightarrow r = \frac{1}{2}$
Lines Equal for $r = \frac{1}{2}$

(30) $(x,y) = (6-2r, 5+r)$ and $(x,y) = (4-2r, 5+2r)$
These lines are equal if $6-2r = 4-2r$ and $5+r = 5+2r$.
$6-2r = 4-2r \longleftrightarrow 0 = -2$ which is impossible
no solution. These lines are not equal
$5+r = 5+2r \longleftrightarrow r = 0$.

also $(6,5)$ is on the first line $(x,y)=(6-2r,\ 5+r)$ for $r=0$.

Indirectly, assume $(6,5)$ is on the second line $(x,y)=(4-2r,\ 5+2r)$

$$6=4-2r \iff 2r=-2 \iff r=-1$$

$$5=5+2r \leftrightarrow r=0$$

Clearly $1 \neq 0$ $\therefore$ $(6,5)$ is not on the second line.

$\therefore$ These lines are not equal

Identify the line specified by each equation

(31) $(x,y)=(-3-2r,\ -5+3r)$

$$\mathcal{L}=\{(-3,-5)+r(-2,3)\}$$

$$(x,y)=(-3,-5)=(-2r,\ 3r)$$

$$(x,y)=(-3,-5)+r(-2,3)$$

The line contains $(-3,-5)$ and has as a direction vector $(-2,3)$.

(32) $(x,y)=(2-4r,\ 3+5r)$

$$(x,y)=(2,3)=(-4r,\ 5r)$$

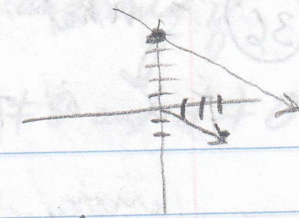
$$(x,y)=(2,3)+r(-4,5)$$

The line contains $(2,3)$ and has as a direction vector $(-4,5)$

(33) $(x,y) = (3r, 5-2r)$

$(x,y) = (0,5) = (3r, -2r)$

$(x,y) = (0,5) + r(3,-2)$



The line contains $(0,5)$ and has $(3,-2)$ as a direction vector.

(34) $(x,y) = (-2r, 3r)$

$(x,y) = (0,0) = (-2r, 3r) = (0,0) + r(-2,3)$

The line contains $(0,0)$ and has $(-2,3)$ as a direction vector.

It passes through the origin.

In exercises 35, 36 let the line

$\mathcal{L} = \{(a,b) + t(c,d)\}$ where

$\|(c,d)\| = 1.$ $\sqrt{a^2+c^2} = 1$ $\therefore a^2+c^2 = 1$

(35) What must be true if the representation of $\mathcal{L}$ in the coordinate plane is the horizontal axis?

$b=0$

The possible unit direction vectors of the horizontal axis are $(1,0)$ and $(-1,0)$

$\therefore \; c = \pm 1, \; d = 0$

(a,b) is a point on the horizontal axis so $b=0$, $a \in \mathbb{R}$

(36) What must be true if the representation of $\mathcal{L}$ in the coordinate plane is the vertical axis?

The possible unit direction vectors of the vertical axis are $(0,1)$ and $(0,-1)$ $\therefore$ $c=0$, $d=\pm 1$.

(a,b) is a point on the vertical axis so $a=0$, $b \in \mathbb{R}$.

(37) Prove: For all real numbers a and b and those real numbers c and d such that $(c,d) \neq \vec{0}$,

$\{(x,y) : (x,y) = (a+tc, b+td)\}$

and $\{(x,y) : dx - cy = ad - bc\}$

identify the same line.

$(x,y) = (a,b) = (tc, td) = t(c,d)$

Since $(c,d) \neq \vec{0}$, $c \neq 0$ and $d \neq 0$?

$(r,s) \in \{(x,y) : (x,y) = (a+tc, b+td)\}$ if and only if $(r,s) = (a+tc, b+td)$, $r = a+tc$ and $s = b+td$. $t = \frac{r-a}{c}$ and $t = \frac{s-b}{d}$

$\frac{r-a}{c} = \frac{s-b}{d} \longleftrightarrow d(r-a) = c(s-b)$

$dr - da = cs - cb \longleftrightarrow dr - cs = da - cb$

$\therefore$ (r,s) satisfies the second condition if and only if (r,s) satisfies the first condition $c \neq 0$ and $d \neq 0$.

(38) Let P be a point and $\vec{v}$ a nonzero vector. Prove that through P there is one and only one line whose direction vector is $\vec{v}$.

By the definition of a line (p 170 in text Dolciani MIA):
$\mathcal{L}$ is called the line through P with direction vector $\vec{v}$, where $\mathcal{L}$ is a set of points $X(x,y)$ in $R \times R$.
We write $\mathcal{L} = \{P + t\vec{v}\}$ and call $X = P + t\vec{v}$ a vector equation of the line.

So, given P and $\vec{v}$, there exists at least one such line.

To prove there is only one such line, assume that there is more than one line with direction vector $\vec{v}$ through $P(x_0, y_0)$.
Let $\mathcal{L} = \{P + t\vec{v}\}$ and $\mathcal{S} = \{P + m\vec{v}\}$.
$P = (x_0, y_0)$, $\vec{v} = (v_1, v_2)$, and $m \in R$.
Since t and m can be chosen as any real numbers when $t = m$, $P + t\vec{v} = P + m\vec{v}$

$\therefore$ $\mathcal{L} = \mathcal{S}$

5-3 Exercises : Coincident and Parallel Lines

For each line $\mathcal{L}$ name a direction vector with integral entries.

① $\mathcal{L} = \{(2,-1) + t(-\frac{1}{2}, 3)\}$

$\vec{w} = 2(-\frac{1}{2}, 3) = (-1, 6)$

Another method:

$t(-\frac{1}{2}, 3) = \frac{t}{2}(-1, 6)$; $\vec{w} = (-1, 6)$

(In exercises 1-6 if (w_1, w_2) is an integral direction vector, $(n \cdot w_1, n w_2)$, where n is a nonzero integer, is also.)

② $\mathcal{L} = \{(-2,0) + t(\frac{1}{3}, -\frac{3}{2})\}$

$t(\frac{1}{3}, -\frac{3}{2}) = \frac{t}{6}(2, -9)$; $\vec{w} = (2, -9)$

③ $\mathcal{L} = \{(3+2t, 5+2t)\}$

$= \{(3,5) + t(2,2)\}$

$\vec{w} = (2,2)$

④ $\mathcal{L} = \{(6 - \frac{1}{2}t, 4 + \frac{2}{3}t)\}$

$= \{(6,4) + t(-\frac{1}{2}, \frac{2}{3})\}$

$t(-\frac{1}{2}, \frac{2}{3}) = \frac{t}{6}(-3, 4)$; $\vec{w} = (-3, 4)$

(5) $\mathcal{L} = \{(-3t, 6 + \frac{1}{2}t)\} = \{(0,6) + t(-3, \frac{1}{2})\}$

$t(-3, \frac{1}{2}) = \frac{t}{2}(-6, 1)$; $\vec{w} = (-6, 1)$

(6) $\mathcal{L} = \{(5 + \frac{1}{3}t, \frac{3}{2}t)\} = \{(5,0) + t(\frac{1}{3}, \frac{3}{2})\}$

$t(\frac{1}{3}, \frac{3}{2}) = \frac{t}{6}(2, 9)$; $\vec{w} = (2, 9)$

If in the plane $\mathcal{L}$ and $\mathcal{M}$ are lines having $\vec{v}$ as direction vector and containing points P and T, respectively, determine whether $\mathcal{L}$ and $\mathcal{M}$ coincide.

(7) $P(0,4)$; $T(-2,1)$; $\vec{v} = (2,3)$

$[(0,4) - (-2,1)] \cdot (2,3)_p = (2,3) \cdot (-3,2)$

$= -6 + 6 = 0$ $\therefore$ $\mathcal{L}$ and $\mathcal{M}$ coincide.

(8) $P(5,0)$, $T(3,4)$; $\vec{v} = (-1,2)$

$[(5,0) - (3,4)] \cdot (-1,2)_p = (2,-4) \cdot (-2,-1)$

$= -4 + 4 = 0$ $\therefore$ $\mathcal{L}$ and $\mathcal{M}$ coincide

(9) $P(1,2)$; $T(-1,4)$; $\vec{v} = (-2,3)$

Formally, $\mathcal{L} = \{P + t\vec{v}\} = \{(1,2) + t(-2,3)\}$

$\mathcal{M} = \{T + s\vec{v}\} = \{(-1,4) + s(-2,3)\}$

$\mathcal{L}$ and $\mathcal{M}$ coincide if $(P - T) \cdot \vec{v}_p = 0$

$[(1,2) - (-1,4)] \cdot (-2,3)_p = (2,-2) \cdot (-3,-2)$

$= -6 + 4 = -2 \neq 0$ $\therefore$ $\mathcal{L}$ and $\mathcal{M}$ do not coincide.

We say $\mathcal{L}$ and $\mathcal{M}$ are DISTINCT.

(10) $P(-2,1)$; $T(-5,6)$; $\vec{v} = (-3,5)$

$$[(-2,1)-(-5,6)]\cdot(-3,5)_p$$

$$= (3,-5)\cdot(-5,-3) = -15+15 = 0$$

$\therefore$ $\mathcal{L}$ and $\mathcal{M}$ coincide

(11) $P(3,1)$; $T(4,-2)$; $\vec{v} = (-1,-3)$

$$[(3,1)-(4,-2)]\cdot(-1,-3)_p = (-1,3)\cdot(3,-1)$$

$= -3-3 = -6 \neq 0$ $\therefore$ $\mathcal{L}$ and $\mathcal{M}$ are distinct.

(12) $P(-1,-3)$; $T(-7,1)$; $\vec{v} = (3,-2)$

$$[(-1,-3)-(-7,1)]\cdot(3,-2)_p$$

$$= (6,-4)\cdot(2,3) = 12-12 = 0$$

$\therefore$ $\mathcal{L}$ and $\mathcal{M}$ coincide

In each exercise determine whether $\mathcal{L}$ and $\mathcal{M}$ are parallel lines, and, if parallel, whether they coincide.

(13) $\mathcal{L} = \{(5-3t, -2+2t)\}$ and
$\mathcal{M} = \{(2-6s, -1-4s)\}$

In exercises 13–18, $\mathcal{L}$ and $\mathcal{M}$ are parallel if the dot product of the direction vector of $\mathcal{L}$, $\vec{w}$, and the perpendicular vector, $\vec{v}_p$, to the direction vector $\vec{v}$ of $\mathcal{M}$ is zero; that is, $\vec{w} \cdot \vec{v}_p = 0$.

If $\mathcal{L}$ and $\mathcal{M}$ are parallel, they are distinct if there is a point $X \in \mathcal{L}$ such that $X \notin \mathcal{M}$, that is, $(X - P_m) \cdot \vec{v}_p \neq 0$.
(P_m denotes a point in $\mathcal{M}$).

So, for (13) $\mathcal{L} = \{(5,-2) + t(-3,2)\}$

$\vec{w} = (-3,2)$

$\mathcal{M} = \{(2,-1) + s(-6,-4)\}$; $\vec{v} = (-6,-4)$

$\vec{v}_p = (4,-6)$

$$\vec{w} \cdot \vec{v}_p = (-3,2) \cdot (4,-6) = -12 - 12 = -24 \neq 0$$

So, $\mathcal{L}$ and $\mathcal{M}$ are not parallel.

(14) $\mathcal{L} = \{(2+t, -3-3t)\} = \{(2,-3) + t(1,-3)\}$

$\mathcal{M} = \{(5-3s, 2+9s)\} = \{(5,2) + s(-3,9)\}$

direction vector of $\mathcal{L}$ is $\vec{w} = (1,-3)$

direction vector of $\mathcal{M}$ is $\vec{v} = (-3,9)$ so $\vec{v}_p = (-9,-3)$

$\vec{w} \cdot \vec{v}_p = (1,-3) \cdot (-9,-3) = -9 + 9 = 0$

so $\mathcal{L}$ and $\mathcal{M}$ are parallel

$\mathcal{L}$ and $\mathcal{M}$ are parallel and are distinct if there is a point $X \in \mathcal{L}$ such that $X \notin \mathcal{M}$; i.e. $(X - P_m) \cdot \vec{V}_p \neq 0$.

Point $(2,-3) \in \mathcal{L}$

Point $(5,2) \in \mathcal{M}$

To see if $(2,-3)$ is in $\mathcal{M}$:

$$[(2,-3)-(5,2)] \cdot (-9,-3) = (-3,-5)\cdot(-9,-3)$$

$= 27 + 15 = 42 \neq 0$ $\therefore$ $\mathcal{L}$ and $\mathcal{M}$ are distinct lines, that is, they do not coincide.

$\therefore$ $\mathcal{L}$ and $\mathcal{M}$ are distinct parallel lines.

(15) $\mathcal{L} = \{(4 - \frac{1}{2}t, 1 + \frac{2}{3}t)\}$
$\mathcal{M} = \{(3+s, -2 + \frac{4}{3}s)\}$

$\mathcal{L} = \{(4,1) + t(-\frac{1}{2}, \frac{2}{3})\}$, $\vec{W} = (-\frac{1}{2}, \frac{2}{3})$
$\mathcal{M} = \{(3,-2) + t(1, \frac{4}{3})\}$ $\vec{V} = (1, \frac{4}{3})$
$\vec{V}_p = (-\frac{4}{3}, 1)$
$\vec{W} \cdot \vec{V}_p = (-\frac{1}{2}, \frac{2}{3}) \cdot (1, \frac{4}{3}) = -\frac{1}{2} + \frac{8}{9} \neq 0$
$\therefore$ $\mathcal{L}$ and $\mathcal{M}$ are not parallel

(16) $\mathcal{L} = \{(\frac{2}{3}t, 5+t)\}$ and $\mathcal{M} = \{(2-s, -\frac{3}{2}s)\}$

Are these lines parallel?

$\mathcal{L} = \{(0,5) + t(\frac{2}{3}, 1)\}$ so $\vec{w} = (\frac{2}{3}, 1)$

$\mathcal{M} = \{(2,0) + s(-1, \frac{3}{2})\}$ so $\vec{v}_p = (\frac{3}{2}, -1)$

$\vec{w} \cdot \vec{v}_p = (\frac{2}{3}, 1) \cdot (\frac{3}{2}, -1) = 1 - 1 = 0$

$\therefore$ $\mathcal{L}$ and $\mathcal{M}$ are parallel

$$(X - P_m) \cdot \vec{v}_p = [(0,5) - (2,0)] \cdot (\tfrac{3}{2}, -1)$$
$$= (-2, 5) \cdot (\tfrac{3}{2}, -1) = -3 - 5 \neq 0$$

$\therefore$ $\mathcal{L}$ and $\mathcal{M}$ are distinct parallel lines

(17) $\mathcal{L} = \{(t, -\frac{3}{5}t)\}$

$\mathcal{M} = \{(-3 + \frac{2}{3}s, 6 + \frac{2}{5}s)\}$

$\mathcal{L} = \{(0,0) + t(1, -\frac{3}{5})\}$ so $\vec{w} = (1, -\frac{3}{5})$

$\mathcal{M} = \{(-3,6) + t(\frac{2}{3}, \frac{2}{5})\}$ so $\vec{v}_p = (-\frac{2}{5}, \frac{2}{3})$

$\vec{w} \cdot \vec{v}_p = (1, -\frac{3}{5}) \cdot (-\frac{2}{5}, \frac{2}{3}) = -\frac{2}{5} - \frac{6}{15} \neq 0$

$\therefore$ $\mathcal{L}$ and $\mathcal{M}$ are not parallel

(18) $\mathcal{L} = \{(5+\sqrt{3}\,t,\ -2-\sqrt{2}\,t)\}$

$\mathcal{M} = \{(-1-\sqrt{6}\,t,\ 5+2t)\}$

$\mathcal{L} = \{(5,-2) + t(\sqrt{3}, -\sqrt{2})\}$; $\vec{w} = (\sqrt{3}, -\sqrt{2})$

$\mathcal{M} = \{(-1,5) + s(-\sqrt{6}, 2)\}$; $\vec{v}_p = (-2, -\sqrt{6})$

$$\vec{w} \cdot \vec{v}_p = (\sqrt{3}, -\sqrt{2}) \cdot (-2, -\sqrt{6})$$

$$= -2\sqrt{3} + \sqrt{12} = -2\sqrt{3} + 2\sqrt{3} = 0$$

$\therefore$ $\mathcal{L}$ and $\mathcal{M}$ are parallel

Are they distinct lines?

$$(X - P_m) \cdot \vec{v}_p = [(5,-2) - (-1,5)] \cdot (-2, -\sqrt{6})$$

$$= (6,-7) \cdot (-2, -\sqrt{6}) = -12 + 7\sqrt{6} \neq 0$$

$\mathcal{L}$ and $\mathcal{M}$ are distinct parallel lines

Find two nonzero vectors $\vec{v}$ such that

$T \in \{P + t\vec{v}\}$

(19) $T(2,3)$; $P(6,-4)$

$T-P = (-4,7) = t\vec{v}$

$t=1,\ \vec{v} = (-4,7)$; $t=-1,\ \vec{v} = (4,-7)$

(20) $T(-3,5)$; $P(2,6)$; $T-P = (-5,-1) = t\vec{v}$

$t=1$: $\vec{v} = (-5,-1)$; $t=-1$: $\vec{v} = (5,-1)$

Note: In order for $T \in \{P + t\vec{v}\}$, the equation

$T = P + t\vec{v}$ must be true

$T - P = t\vec{v}$

We are simply finding vectors $(T-P)$ and $-(T-P)$.

(21) $T(4,0)$; $P(3,-5)$; $T-P = (1,5) = t\vec{v}$

$t=1$: $\vec{v} = (1,5)$; $t=-1$: $\vec{v} = (-1,-5)$

(22) $T(4,3)$; $P(0,-2)$; $T-P = (4,5) = t\vec{v}$

$t=1$: $\vec{v} = (4,5)$; $t=-1$: $\vec{v} = (-4,-5)$

(23) If $(2,-3) \in \{(a,2a) + t(1,3)\}$, find a.

$[(2,-3) - (a,2a)] \cdot (1,3)_p = 0$

$(2-a, -3-2a) \cdot (-3,1) = -3(2-a) + (-3-2a) = 0$

$-6 + 3a - 3 - 2a = 0 \leftrightarrow -9 + a = 0 \leftrightarrow a = 9$

(24) If $(2a,-3a) \in \{(4,3) + t(-1,3)\}$, find a.

$[(2a,-3a) - (4,3)] \cdot (-1,3)_p = 0$

$(2a-4, -3a-3) \cdot (-3,-1) = -3(2a-4) - (-3a-3)$

$= -6a + 12 + 3a + 3 = -3a + 15 = 0 \leftrightarrow a = 5$

(25) Let $\mathcal{L}$ be a line and $\vec{w}$ a nonzero vector which is not a direction vector of $\mathcal{L}$.

Prove that for some scalar r, $r\vec{w} \in \mathcal{L}$.

Solution:

$$\mathcal{L} = \{P + t\vec{v}\}, \quad P = (x_0, y_0)$$

$$\vec{v} = (v_1, v_2), \quad \vec{v_p} = (-v_2, v_1)$$

$$\vec{w} = (w_1, w_2), \quad r\vec{w} = (rw_1, rw_2)$$

$r\vec{w} \in \mathcal{L}$ if $(r\vec{w} - P)\cdot\vec{v_p} = 0$,

if $(rw_1 - x_0, rw_2 - y_0)\cdot(-v_2, v_1) = 0$

$$-v_2(rw_1 - x_0) + v_1(rw_2 - y_0) = 0$$

$$-v_2 r w_1 + v_2 x_0 + v_1 r w_2 - v_1 y_0 = 0$$

$$r(w_2 v_1 - w_1 v_2) = v_1 y_0 - v_2 x_0$$

$$r = \frac{v_1 y_0 - v_2 x_0}{w_2 v_1 - w_1 v_2}$$

($w_2 v_1 - w_1 v_2 \neq 0$ since $\vec{w}$ and $\vec{v}$ are nonzero and not the same vector)

$\therefore$ if $r = \dfrac{v_1 y_0 - v_2 x_0}{w_2 v_1 - w_1 v_2}$ then $r\vec{w} \in \mathcal{L}$

(26) Show that the line $\mathcal{L} = \{P + t(S-P)\}$ can also be denoted by the equation $\mathcal{L} = \{(1-t)P + tS\}$.

Use this result to show that $P \in \mathcal{L}$ and $S \in \mathcal{L}$.

$P + t(S-P) = P + tS - tP = P(1-t) + tS$

Let $t = 1$ in $P + t(S-P)$, then $P + 1(S-P) = P + S - P$ $= S \therefore S \in \mathcal{L}$

Let $t = 0$ in $(1-t)P + tS$, then $(1-0)P + (0)S$ $= P \therefore P \in \mathcal{L}$

(27) Prove that if $\mathcal{L}$ and $\mathcal{M}$ are parallel lines, and $\mathcal{L} \neq \mathcal{M}$, then $\mathcal{L} \cap \mathcal{M} = \emptyset$.

Given $\mathcal{L} \parallel \mathcal{M}$ and $\mathcal{L} \neq \mathcal{M}$.

Suppose $\mathcal{L} \cap \mathcal{M} \neq \emptyset$; then some point X lies on both lines.

If $X \in \mathcal{L}$ and $X \in \mathcal{M}$, this contradicts $\mathcal{L} \parallel \mathcal{M} \therefore \mathcal{L} \cap \mathcal{M} = \emptyset$

(28) Prove that if $\mathcal{L}$ and $\mathcal{M}$ are lines, and $\mathcal{L}$ and $\mathcal{M}$ are not parallel, then $\mathcal{L} \cap \mathcal{M}$ contains just one point.

Note: This is the last problem of 5-3, and it certainly required access to Solution Key ISBN 0.395-25555-4, 978-039525555 1

OK. Here it is.

(28) Prove that if $\mathcal{L}$ and $\mathcal{M}$ are lines, and $\mathcal{L}$ and $\mathcal{M}$ are not parallel, then $\mathcal{L} \cap \mathcal{M}$ contains just one point.

$\boxed{\text{SOLUTION}}$

Let $\mathcal{L} = \{P + t\vec{v}\}$, $P = (a, b)$, $\vec{v} = (v_1, v_2)$, $\mathcal{M} = \{S + r\vec{w}\}$, $S = (c, d)$, $\vec{w} = (w_1, w_2)$ where $\vec{v}$ is not parallel to $\vec{v}$, $\vec{v}$ and $\vec{w}$ are nonzero vectors, P and S are distinct points, $t \in \mathbb{R}$, $r \in \mathbb{R}$.

To show there is at least one point $T(x, y)$ in $\mathcal{L}$ and in $\mathcal{M}$; Suppose $T \in \mathcal{L}$; then $(x, y) = (a, b) + t_1(v_1, v_2)$ for some $t_1 \in \mathbb{R}$.

$x = a + t_1 v_1$, $y = b + t_1 v_2$

$T \in \mathcal{M}$ if $(a + t_1 v_1, b + t_1 v_2) = (c, d) + r_1(w_1, w_2)$ for some $r_1 \in \mathbb{R}$, or

$(a + t_1 v_1, b + t_1 v_2) - (c, d) = r_1(w_1, w_2)$,

$(a + t_1 v_1 - c, b + t_1 v_2 - d) = (r_1 w_1, r_1 w_2)$,

$a + t_1 v_1 - c = r_1 w_1$ and $b + t_1 v_2 - d = r_1 w_2$,

or $r_1 = \dfrac{a + t_1 v_1 - c}{w_1}$ and $r_1 = \dfrac{b + t_1 v_2 - d}{w_2}$

or,

or, $\dfrac{a + t_1 v_1 - c}{w_1} = \dfrac{b + t_1 v_2 - d}{w_2}$,

$$a w_2 + t_1 (v_1 w_2) - c w_2 = b w_1 + t_1 (v_2 w_1) - d w_1 ,$$

$$t_1 (v_1 w_2 - v_2 w_1) = b w_1 + c w_2 - d w_1 - a w_2$$

$$t_1 = \frac{b w_1 + c w_2 - d w_1 - a w_2}{v_1 w_2 - v_2 w_1} \quad \left(\begin{array}{c} v_1 w_2 - v_2 w_1 \neq 0 \\ \text{since } \vec{v}_1 \nparallel \vec{w} \end{array} \right)$$

For this particular t_1, $T \in \mathcal{M}$.

Since $T \in \mathcal{L}$ and $T \in \mathcal{M}$, $\mathcal{L} = \{ T + t\vec{v} \}$ and $\mathcal{M} = \{ T + r\vec{w} \}$.

Suppose $Q \neq T$, $Q \in \mathcal{L}$ and $Q \in \mathcal{M}$.
For some $t_2 \neq 0$, $Q = T + t_2 \vec{v}$
and for some $r_2 \neq 0$, $Q = T + r_2 \vec{w}$.

$\therefore$ $T + t_2 \vec{v} = T + r_2 \vec{w}$, $t_2 \vec{v} = r_2 \vec{w}$,

$\vec{v} = \dfrac{r_2}{t_2} \vec{w}$. But $\vec{v} \parallel \vec{w}$. $\boxed{\text{Given: } \vec{v} \nparallel \vec{w}}$

The assumption that a second point Q is in $\mathcal{L}$ and $\mathcal{M}$ leads to a contradiction, and so there can be only one point T in common.

$\boxed{5\text{-}4}$ The Line Through Two points

Determine a vector equation for the line $\mathcal{L}$ passing through S and T.

① $S(2,3)$; $T(4,6)$

$\mathcal{L} = \{S + t(S-T)\} = \{(2,3) + t(-2,-3)\}$

$(x,y) = (2-2t,\ 3-3t)$

$\mathcal{L} = \{S + t(T-S)\}$

or $T-S = (4,6)-(2,3) = (2,3)$

$\mathcal{L} = \{(2,3) + t(2,3)\}$

$(x,y) = (2+2t,\ 3+3t)$

② $S(0,5)$; $T(3,0)$

$T-S = (3,0)-(0,5) = (3,-5)$

$\mathcal{L} = \{(0,5) + t(3,-5)\}$

$(x,y) = (3t,\ 5-5t)$

③ $S(-3,1)$; $T(2,-5)$

$T-S = (2-(-3),\ -5-1) = (5,-6)$

$\mathcal{L} = \{(-3,1) + t(5,-6)\}$

$(x,y) = (-3+5t,\ 1-6t)$

or $S-T = (-3-2,\ 1-(-5)) = (-5,6)$

$\mathcal{L} = \{(2,-5) + t(-5,6)\}$

$(x,y) = (2-5t,\ -5+6t)$

There are, of course, more than one representation. Generally $\mathcal{L} = \{ P + t(X-P) \}$ where $\mathcal{L}$ passes through points X and P.

④ $S(3,-2)$; $T(-4,-2)$

$S-T = (3-(-4), -2-(-2)) = (7,0)$

$\mathcal{L} = \{ (-4,-2) + t(7,0) \}$

$(x,y) = (-4+7t, -2)$

⑤ $S(5,1)$; $T(2,-3)$

$S-T = (5-2, 1-(-3)) = (3,4)$

$\mathcal{L} = \{ (2,-3) + t(3,4) \}$

$(x,y) = (2+3t, -3+4t)$

⑥ $S(0,0)$; $T(a,b)$

$T-S = (a,b)$

$\mathcal{L} = \{ (0,0) + t(a,b) \}$

$(x,y) = (a \cdot t, bt)$

⑦ $S(a,0)$, $T(0,a)$

$S-T = (a,-a)$

$\mathcal{L} = \{ (0,a) + t(a,-a) \}$

$(x,y) = (a, a-at)$

⑧ $S(\sqrt{3},1)$. $T(2\sqrt{3},-4)$

$T-S = (\sqrt{3},-5)$. $\mathcal{L} = \{ (\sqrt{3},1) + t(\sqrt{3},-5) \}$

$(x,y) = (\sqrt{3}+\sqrt{3}t, 1-5t)$

Show that PS is parallel to RT

(9) $P(1,2)$; $S(-2,4)$; $R(3,-6)$; $T(-3,-2)$

$P-S = (1-(-2), 2-4) = (3,-2) = \vec{v}$

$R-T = (3-(-3), -6-(-2)) = (6,-4) = 2\vec{v}$

Hence $(P-S) \parallel (R-T)$

Since PS and RT have parallel direction vectors, they are parallel.

(10) $P(0,-1)$; $S(-5,2)$; $R(7,0)$; $T(-3,6)$

$P-S = (0-(-5), -1-2) = (5,-3) = \vec{v}$

$R-T = (7-(-3), 0-6) = (10,-6) = 2\vec{v}$

Hence, $(P-S) \parallel (R-T)$

Since PS and RT have parallel direction vectors, they are parallel.

Write a vector equation of the line $\mathcal{L}$ through M parallel to PS.

(11) $M(2,3)$; $P(1,4)$; $S(3,5)$

$\mathcal{L} = \{(2,3) + t[(1,4)-(3,5)]$

$= \{(2,3) + t(-2,-1)\}$

$(x,y) = (2-2t, 3-t)$

(12) $M(0,1);\ P(-2,3);\ S(2,0)$

$$\mathcal{L} = \{(0,1) + t[(2,0)-(-2,3)]\}$$
$$= \{(0,1) + t(4,-3)\}$$
$$(x,y) = (4t, 1-3t)$$

notice $P-S = (-2-2, 3) = (-4,3) = \vec{v}$

[PS implies P-S]*

$$\mathcal{L} = \{(0,1) + t(-4,3)\}$$
$$(x,y) = (-4, 1+3t)$$

(13) $M(2,-3),\ P(3,-1),\ S(-1,3)$

$P-S = (3,-1)-(-1,3) = (4,-4) = 4(1,-1);$

$(1,-1) = \vec{v}$

$$\mathcal{L} = \{(2,-3) + t(1,-1)\}$$
$$(x,y) = (2+t, -3-t)$$

(14) $M(0,0),\ P(3,5),\ S(-2,6)$

$P-S = (3-(-2), 5-6) = (5,-1) = \vec{v}$

$$\mathcal{L} = \{M + t(P-S)\} = \{(0,0) + t(5,-1)\}$$
$$(x,y) = (5t, -t)$$

(15) $M(3,6),\ P(-1,2),\ S(4,-1)$

$P-S = (-1-4, 2-(-1)) = (-5,3) = \vec{v}$

$$\mathcal{L} = \{M + t\vec{v}\} = \{(3,6) + t(-5,3)\}$$
$$(x,y) = (3-5t, 6+3t)$$

(16) $M(4,3)$; $P(2,-1)$; $S(1,-2)$

$P-S = (2-1, -1-(-2)) = (1,1) = \vec{v}$

$\mathcal{L} = \{ (4,3) + t(1,1) \}$

$(x,y) = (4+t, 3+t)$

Determine an equation of the ray PQ having P as endpoint and containing the point Q.

(17) $P(0,0)$; $Q(5,-1)$

$Q-P = (5,-1) = \vec{v}$

$\mathcal{L} = \{ (0,0) + t(5,-1) \}$

$(x,y) = (5t, -t)$, $t \geq 0$

(18) $P(0,0)$; $Q(-2,-4)$

$Q-P = (-2,-4) = 2(-1,-2) = 2\vec{v}$

$\mathcal{L} = \{ (0,0) + t(-1,-2) \}$

$(x,y) = (-t, -2t)$, $t \geq 0$

(19) $P(3,-1)$; $Q(0,0)$

$Q-P = (-3, 1) = \vec{v}$

$\mathcal{L} = \{ (3,-1) + t(-3,1) \}$

$(x,y) = (3-3t, -1+t)$, $t \geq 0$

(20) $P(-2,-4)$; $Q(1,6)$

$Q-P=(1-(-2),\ 6-(-4))=(3,10)=\vec{v}$

$\mathcal{L}=\{(-2,-4)+t(3,10)\}$

$(x,y)=(-2+3t,\ -4+10t)$; $t\geq 0$

Determine whether the given line M coincides with the line PQ.

(21) $M=\{(6+t,\ 2-3t)\}$; $P(6,2)$; $Q(4,-4)$

$P-Q=(6-4,\ 2-(-4))=(2,6)=2(1,3)$

$M=\{(6,2)+t(1,-3)\}$

Both PQ and M pass through $(6,2)$, but are not coincident since their direction vectors are not parallel: $(1,3)\neq k(1,-3)$ for some k.

(22) $M=\{(2-3t,\ -3+2t)\}$; $P(5,-5)$; $Q(-4,3)$

$M=\{(2,-3)+t(-3,2)\}$, $\vec{v}=(-3,2)$

$P-Q=(5-(-4),\ -5-3)=(9,-8)=\vec{w}$

$\vec{v}\neq k\vec{w}$ for any k. $\therefore$ M and PQ do not coincide.

(23) $M = \{(3,-1) + t(-1,2)\}$; $P(2,1)$; $Q(5,-5)$

Does line M coincide with the line PQ?

$\vec{v} = (-1,2)$

$P - Q = (-3,6) = 3(-1,2) = 3\vec{v}$ (yes)

(24) $M = \{(-2,4) + t(2,-3)\}$; $P(0,3)$; $Q(-2,4)$

$\vec{v} = (2,-3)$

$P - Q = (2,-1) = \vec{w}$

$\vec{v} \neq k\vec{w}$ for any k $\therefore$ M and PQ do not coincide. They are not even parallel.

If P, S, and R are distinct points in the plane, which of the following permit you to conclude that the points are collinear?

(25) $P \in SR$

SR identifies the line containing both S and R. If $P \in SR$, then SR contains P, S, R, and they are collinear.

(26) distance (SP) = dist (PR)

This says nothing about the directions of SP and PR. The three points could form the vertices of an isosceles triangle.

(27) $R - P = t(S - P)$

This implies $R-P$ and $S-P$ are parallel vectors

By the theorem (p 180 MIA), the points P, R, and S in the plane are collinear if and only if $R-P$ and $S-P$ are parallel vectors.

(28) $S + P = R$

While it is tempting to plug in values $(1,1)+(2,2)=(3,3)$ and conclude S, P, R are collinear points, this is not necessarily true. $S+P=R$ says nothing about the direction vectors SP, PR, and SR, $\therefore$ nothing can be concluded about collinearity of the points.

Find the value of k for which S, T, and P will be collinear.

(29) $S(1,-1)$; $T(3,1)$; $P(k,4)$

P, S, and T in the plane are collinear if and only if $S-P$ and $T-P$ are parallel vectors.

$S - P = (1,-1) - (k,4) = (1-k, -5) = \vec{v}$

$T - P = (3,1) - (k,4) = (3-k, -3) = \vec{w}$

$(S-P)\cdot(T-P)_p = (1-k, -5)\cdot(3, 3-k) = 0$

$3(1-k) - 5(3-k) = 0 \Leftrightarrow 3 - 3k - 15 + 5k = 0$

$\Leftrightarrow 2k = 12 \Leftrightarrow k = 6$

(30) $S(0,2)$; $T(1,1)$; $P(-1,k)$

$$(S-P)\cdot(T-P)_p = 0$$

$$[(0,2)-(-1,k)]\cdot[(1,1)-(-1,k)]_p = 0$$

$$(1, 2-k)\cdot(2, 1-k)_p = 0$$

$$(1, 2-k)\cdot(k-1, 2) = k-1+2(2-k) = 0$$

$$k-1+4-2k = 0 \leftrightarrow -k+3 = 0$$

$$k = 3$$

(31) $S(4,6)$, $T(-3,8)$, $P(-k,k)$

$$(S-P)\cdot(T-P)_p = 0$$

$$[(4,6)-(-k,k)]\cdot[(-3,8)-(-k,k)] = 0$$

$$(4+k, 6-k)\cdot(-3+k, 8-k)_p = 0$$

$$(4+k, 6-k)\cdot(k-8, -3+k) = 0$$

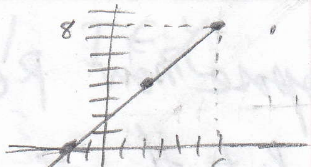
$$(4+k)(k-8)+(6-k)(-3+k) = 0$$

$$4(k-8)+k(k-8)+6(-3+k)-k(-3+k) = 0$$

$$4k-32+k^2-8k-18+6k+3k-k^2 = 0$$

$$5k-50 = 0 \leftrightarrow 5k = 50 \leftrightarrow k = 10$$

(32) $S(2,4);\ T(6,k);\ P(-2,0)$



$(S-P)\cdot(T-P)_p = 0$

$[(2,4)-(-2,0)]\cdot[(6,k)-(-2,0)]_p = 0$

$(4,4)\cdot(8,k)_p = 0$

$(4,4)\cdot(-k,8) = -4k+32 = 0$

$-4k = -32 \longleftrightarrow k = 8$

(33) Prove: If P is any point and $\mathcal{M}$ any line in the plane, one and only one line in the plane passes through P parallel to $\mathcal{M}$.

Let $\mathcal{M} = \{P + t\vec{v}\}$

There are two cases: $P \in \mathcal{M}$ and $P \notin \mathcal{M}$.

(1) Let $P \in \mathcal{M}$

Since $\vec{v}\cdot\vec{v}_p = 0$ for any vector $\vec{v}$, $\mathcal{M} \parallel \mathcal{M}$

$\therefore$ there is at least one parallel.

By theorem on p 177 (MIA) lines having a point in common coincide if and only if they are parallel. Hence, if $P \in \mathcal{L}$ and $P \in \mathcal{M}$ and $\mathcal{L} \parallel \mathcal{M}$, then $\mathcal{L} = \mathcal{M}$ $\therefore$ there is exactly one parallel.

(2) Let $P \notin \mathcal{M}$

For $t=0$ and $t=1$, T and $T+\vec{v}$ are points of $\mathcal{M}$. $\mathcal{L} = \{P + r[T-(T+\vec{v})]\} = \{P + r\vec{v}\}$ has the properties $P \in \mathcal{L}$ and, since $\vec{v}\cdot\vec{v}_p = 0$, $\mathcal{L} \parallel \mathcal{M}$ $\therefore$ there is at least one parallel.

Suppose now $P \in \mathcal{N}$ and $\mathcal{N} \parallel \mathcal{M}$.

But then $\vec{v}$ is a direction vector of $\mathcal{N}$, and so $\mathcal{N} \parallel \mathcal{L}$. $\mathcal{N} = \mathcal{L}$ again by the theorem stated on p. 177.

$\therefore$ there is exactly one parallel in either case.

(34) Prove the second theorem stated on page 180

The points P, T, and S in the plane are collinear if and only if $T-P$ and $S-P$ are parallel vectors.

(1) Assume P, T, and S are collinear.

Then $S \in PT$ and $PT \in \{P + t(T-P)\}$

$\therefore$ $S = P + t_1(T-P)$ for some t_1,

and $S - P = t_1(T-P)$ which is the defining condition for $S-P \parallel T-P$.

(2) Assume $S-P \parallel T-P$.

Then $S-P = t_2(T-P)$ for some real number t_2, and $S = P + t_2(T-P)$ which is the condition that $S \in \mathcal{L}$ for $\mathcal{L} = \{P + t(T-P)\}$

Also, for $t=0$, $P \in \mathcal{L}$ and for $t=1$, $T \in \mathcal{L}$

$\therefore$ S, T, and P are collinear.

(35) Prove: If $P(x,y)$ lies on the line through $S(x_1, x_2)$ and $R(x_2, y_2)$, then
$x = x_1 + t(x_2 - x_1)$ and $y = y_1 + t(y_2 - y_1)$.

Given $P \in RS$.
$RS = \{S + t(R-S)\}$ $\therefore$ $P(x,y) = (x_1, y_1) + t[(x_2, y_2) - (x_1, y_1)]$ for some t,

$(x,y) = (x_1 + t(x_2 - x_1), y_1 + t(y_2 - y_1))$;

or $x = x_1 + t(x_2 - x_1)$ and $y = y_1 + t(y_2 - y_1)$

(36) Let T and S be points other than P belonging to the respective rays PT and PS, each having P as endpoint. Prove that the rays coincide if and only if $T-P$ and $S-P$ are vectors having the same direction.

Ray $PT = \{P + t(T-P), t \geq 0\}$
and ray $PS = \{P + r(S-P), r \geq 0\}$

(repeat) ray $PT = \{P + t(T-P),\ t \geq 0\}$

ray $PS = \{P + r(S-P),\ r \geq 0\}$

(1) Assume the two rays coincide.
Then $P + t(T-P) = P + r(S-P)$, or
$t(T-P) = r(S-P)$.
If $t = 0$, $r = 0$

If $t > 0$, then $r > 0$ and $\frac{r}{t} > 0$,

so that $(T-P) = \frac{r}{t}(S-P)$.

Since $T-P$ is a positive scalar multiple of $S-P$, $T-P$ and $S-P$ have the same direction.

(2) Assuming $T-P$ and $S-P$ have the same direction, $T-P = k(S-P)$, $k \geq 0$.

$$\text{Ray } PT = \{P + t(T-P),\ t \geq 0\}$$
$$= \{P + tk(S-P),\ tk \geq 0\}$$
$$= \text{ray } PS, \text{ letting } r = tk.$$

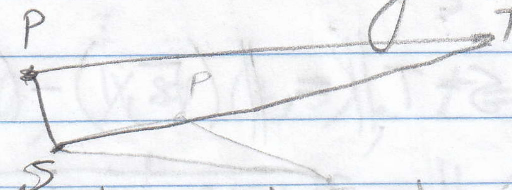
5-5 LINE SEGMENTS

If P, S, and T are distinct points in the plane, which of the following permit you to conclude that P is between S and T?

① $P \in \overleftrightarrow{ST}$.
No. P could be any point on the line ST.

② $P \in \overline{ST}$.
Yes, since P, S, and T are distinct, P must be between the endpoints S and T.

③ $\text{dist}(S, P) < \text{dist}(S, T)$
No, P, S, and T could be the vertices of a scalene triangle.



scalene △
no two sides have same length

④ $\text{dist}(S, P) + \text{dist}(P, T) = \text{dist}(S, T)$

Yes. This is the definition of "between".

Find the length of $\overline{ST}$.

⑤ $S(2, 3)$, $T(-1, 5)$

The length of $\overline{ST} = \text{dist}(S, T) = \|S - T\|$

$$= \sqrt{(2-(-1))^2 + (3-5)^2} = \sqrt{9+4} = \sqrt{13}$$

(6) $S(0,2);\ T(3,-1)$

length $\overline{ST} = \|S-P\| = \|(0,2)-(3,-1)\|$

$= \|(-3,3)\| = \sqrt{(-3)^2+3^2} = \sqrt{18} = 3\sqrt{2}$

(7) $S(1,4);\ T(-2,3)$

length of $\overline{ST} = \|S-T\| = \|(1,4)-(-2,3)\|$

$= \|(3,1)\| = \sqrt{3^2+1^2} = \sqrt{10}$

(8) $S(\sqrt{3},1);\ T(0,4)$

length of $\overline{ST} = \|S-T\| = \|(\sqrt{3},1)-(0,4)\|$

$= \|(\sqrt{3},-3)\| = \sqrt{(\sqrt{3})^2+(-3)^2}$

$= \sqrt{3+9} = \sqrt{12} = 2\sqrt{3}$

(9) $S(3\sqrt{2},\sqrt{3});\ T(\sqrt{2},4\sqrt{3})$

length of $\overline{ST} = \|S-T\| = \|(3\sqrt{2}-\sqrt{2},\sqrt{3}-4\sqrt{3})\|$

$= \|(2\sqrt{2},-3\sqrt{3})\| = \sqrt{(2\sqrt{2})^2+(-3\sqrt{3})^2}$

$= \sqrt{8+27} = \sqrt{35}$

(10) $S(2a, b)$; $T(-a, 3b)$

length of $\overline{ST} = \| S - T \| = \| (2a - (-a), b - 3b) \|$

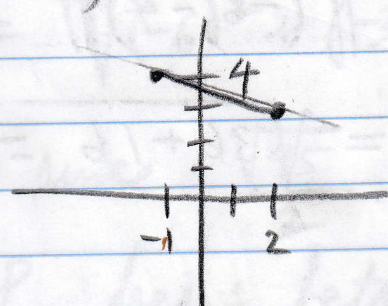
$= \| (3a, -2b) \| = \sqrt{(3a)^2 + (-2b)^2}$

$= \sqrt{9a^2 + 4b^2}$

Represent in the coordinate plane the set of points specified by

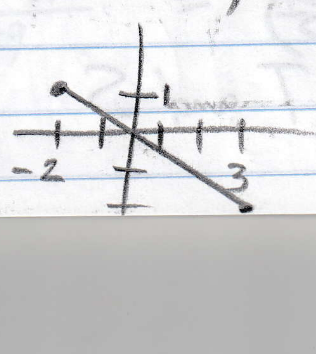
(11) $\{ X : X = (1-t)(2,3) + t(-1,4),$
$0 \leq t \leq 1 \}$

$t = 0$, $X = (2,3)$; $t = 1$, $X = (-1,4)$



(12) $\{ X : X = (1-t)(-2,1) + t(3,-2),$
$0 \leq t \leq 1 \}$

$t = 0$, $X = (-2,1)$; $t = 1$, $X = (3,-2)$



In each exercise determine whether P lies between T and S.

(13) $P(3,0)$; $T(0,-2)$; $S(9,4)$

Does $\text{dist}(T,P) + \text{dist}(P,S) = \text{dist}(T,S)$?

$\text{dist}(T,P) = \| T-P \| = \| (-3,-2) \|$

$= \sqrt{(-3)^2+(-2)^2} = \sqrt{13}$

$\text{dist}(T,S) = \| T-S \| = \| (-9,-6) \|$

$= \sqrt{(-9)^2+(-6)^2} = \sqrt{81+36}$

$= \sqrt{117} = 3\sqrt{13}$

$\text{dist}(P,S) = \| P-S \| = \| (-6,-4) \|$

$= \sqrt{(-6)^2+(-4)^2} = \sqrt{36+16} = \sqrt{52}$

$= 2\sqrt{13}$

$\sqrt{13} + 2\sqrt{13} = 3\sqrt{3}$

$\therefore$ Since $\text{dist}(T,P) + \text{dist}(P,S)$

$= \text{dist}(T,S)$

Yes, P lies between T and S

(14) $P(1, \frac{3}{2})$; $T(-2,0)$; $S(4,3)$

$$dist(T,P) = \|T-P\| = \|(-3, -\tfrac{3}{2})\|$$

$$= \sqrt{(-3)^2 + (-\tfrac{3}{2})^2} = \sqrt{9 + \tfrac{9}{4}} = \sqrt{\tfrac{45}{4}}$$

$$= \frac{\sqrt{45}}{2}$$

$$dist(P,S) = \|P-S\| = \|(-3, -\tfrac{3}{2})\|$$

$$= \sqrt{(-3)^2 + (-\tfrac{3}{2})} = \frac{\sqrt{45}}{2}$$

$$dist(T,S) = \|T-S\| = \|(-6,-3)\|$$

$$= \sqrt{(-6)^2 + (-3)^2} = \sqrt{36+9} = \sqrt{45}$$

Yes, since $\frac{\sqrt{45}}{2} + \frac{\sqrt{45}}{2} = \sqrt{45}$

P lies between T and S

(15) $P(2, 9/7)$, $T(5,0)$, $S(-1,3)$

$$dist(T,P) = \sqrt{(5-2)^2 + (-\tfrac{9}{7})^2} = \sqrt{9 + \tfrac{81}{49}}$$

$$= \sqrt{\frac{441+81}{49}} = \frac{\sqrt{522}}{7} = \frac{3\sqrt{58}}{7}$$

In each exercise determine whether P lies between T and S.

(13) $P(3,0)$; $T(0,-2)$; $S(9,4)$

Does $\text{dist}(T,P) + \text{dist}(P,S) = \text{dist}(T,S)$?

$\text{dist}(T,P) = \| T-P \| = \| (-3,-2) \|$

$= \sqrt{(-3)^2+(-2)^2} = \sqrt{13}$

$\text{dist}(T,S) = \| T-S \| = \| (-9,-6) \|$

$= \sqrt{(-9)^2+(-6)^2} = \sqrt{81+36}$

$= \sqrt{117} = 3\sqrt{13}$

$\text{dist}(P,S) = \| P-S \| = \| (-6,-4) \|$

$= \sqrt{(-6)^2+(-4)^2} = \sqrt{36+16} = \sqrt{52}$

$= 2\sqrt{13}$

$\sqrt{13} + 2\sqrt{13} = 3\sqrt{3}$

$\therefore$ Since $\text{dist}(T,P) + \text{dist}(P,S)$

$= \text{dist}(T,S)$

Yes, P lies between T and S

(14) $P(1, \frac{3}{2})$; $T(-2,0)$; $S(4,3)$

$$dist(T,P) = \|T-P\| = \|(-3, -\tfrac{3}{2})\|$$

$$= \sqrt{(-3)^2 + (-\tfrac{3}{2})^2} = \sqrt{9 + \tfrac{9}{4}} = \sqrt{\tfrac{45}{4}}$$

$$= \frac{\sqrt{45}}{2}$$

$$dist(P,S) = \|P-S\| = \|(-3, -\tfrac{3}{2})\|$$

$$= \sqrt{(-3)^2 + (-\tfrac{3}{2})} = \frac{\sqrt{45}}{2}$$

$$dist(T,S) = \|T-S\| = \|(-6,-3)\|$$

$$= \sqrt{(-6)^2 + (-3)^2} = \sqrt{36+9} = \sqrt{45}$$

Yes, since $\frac{\sqrt{45}}{2} + \frac{\sqrt{45}}{2} = \sqrt{45}$

P lies between T and S

(15) $P(2, 9/7)$; $T(5,0)$; $S(-1,3)$

$$dist(T,P) = \left((5-2)^2 + (-\tfrac{9}{7})^2\right)^{1/2} = \sqrt{9 + \tfrac{81}{49}}$$

$$= \sqrt{\frac{441+81}{49}} = \frac{\sqrt{522}}{7} = \frac{3\sqrt{58}}{7}$$

$$dist(P,S) = \sqrt{(2-(-1))^2 + (\tfrac{9}{7}-3)^2}$$

$$= \sqrt{9 + (\tfrac{30}{7})^2} = \sqrt{9 + \tfrac{900}{49}}$$

$$= \sqrt{\frac{441+900}{49}} = \frac{\sqrt{1341}}{7} = \frac{3}{7}\sqrt{149}$$

$$dist(T,S) = \sqrt{(5-(-1))^2 + (-3)^2}$$

$$= \sqrt{36 + 9} = \sqrt{45}$$

Does $dist(T,P) + dist(P,S) = dist(T,S)$?

$$\frac{3\sqrt{58}}{7} + \frac{3}{7}\sqrt{149} \neq \sqrt{45}$$

$\therefore$ P does not lie between T and S.

(16) $P(a,-2)$; $T(a,3)$; $S(a,0)$

$$dist(T,P) = \|T-P\| = \|(0,5)\|$$

$$= \sqrt{5^2} = 5$$

$$dist(P,S) = \|P-S\| = \|(0,-2)\| = \sqrt{(-2)^2}$$

$$= 2$$

$$dist(T,S) = \|T-S\| = \|(0,3)\| = \sqrt{3^2} = 3$$

No. $5+2 \neq 3$, so P does not lie between T and S

(about #16)

Note that, since $2+3=5$, T lies between P and S.

---

(17) To what subset of numbers must the domain of r be restricted if

$$\{ X : X = (1-3r)P + 3rS \}$$

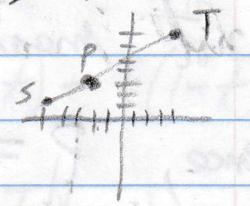
defines the segment $\overline{PS}$?

$3r \geq 0$ and $3r \leq 1$, so $0 \leq r \leq \frac{1}{3}$

Explanation: Let $t = 3r$ and use the definition of $\overline{PS} = \{X : X = (1-t)P + tS,\ 0 \leq t \leq 1\}$

The given set defines $\overline{PS}$ if and only if $0 \leq 3r \leq 1$, that is $0 \leq r \leq \frac{1}{3}$.

(18) In what ratio does $P(-2,3)$ divide the segment joining $S(-5,1)$ and $T(4,7)$?



$$\frac{\text{dist}(P,S)}{\text{dist}(P,T)} = \frac{\|P-S\|}{\|P-T\|}$$

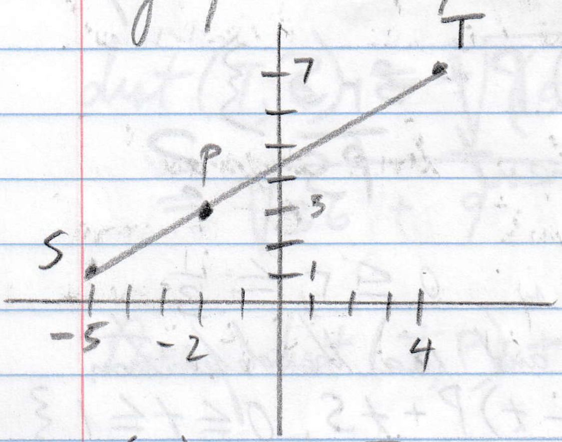
$$= \frac{\|(-2,3)-(-5,1)\|}{\|(-2,3)-(4,7)\|} = \frac{\|(3,2)\|}{\|(-6,-4)\|} = \frac{\sqrt{3^2+2^2}}{\sqrt{(-6)^2+(-4)^2}}$$

$$= \frac{\sqrt{13}}{\sqrt{52}} = \frac{\sqrt{13}}{2\sqrt{13}} = \frac{1}{2}$$

Side note about exercise 18:

$P(-2,3)$ divides the segment joining $S(-5,1)$ and $T(4,7)$

A graphical representation:



$$\text{dist}(S,T) = \|S-T\|$$
$$= \|(-5,1)-(4,7)\|$$
$$= \|(-9,-6)\|$$
$$= \sqrt{(-9)^2+(-6)^2}$$
$$= \sqrt{81+36} = \sqrt{117}$$
$$= 3\sqrt{13}$$

[since $9 \cdot 13 = 117$]

$\text{dist}(P,S) = \sqrt{13}$

$\text{dist}(P,T) = 2\sqrt{13}$

$\dfrac{\text{dist}(P,S)}{\text{dist}(P,T)} = \dfrac{1}{2}$ so P divides the segment at a ratio of $\frac{1}{2}$. Note that this does not mean midpoint. In that case $r=1$.

Hence, $P = \dfrac{1}{1+r} S + \dfrac{r}{1+r} T$

$$P = \frac{1}{1+\frac{1}{2}} S + \frac{\frac{1}{2}}{1+\frac{1}{2}} T = \frac{2}{3} S + \frac{1}{3} T$$

$$P = \frac{2}{3}(-5,1) + \frac{1}{3}(4,7) = \left(\frac{-10}{3}, \frac{2}{3}\right) + \left(\frac{4}{3}, \frac{7}{3}\right)$$

$$= \left(-\frac{6}{3}, \frac{9}{3}\right) = (-2,3)$$

So $\|S-P\| = \frac{1}{2}\|P-T\|$

$P = \frac{2}{3}S + \frac{1}{3}T$

Intuitively, P is twice as far from T as it is to S, and it may be tempting to "feel" that $\frac{2}{3}S$ and $\frac{1}{3}T$ should be reversed.

To locate the point P of a line segment $\overline{ST}$ that partitions $\overline{ST}$ into two segments whose lengths are in prescribed ratio r, note that, for $P = (1-t)S + tT$, $0 \leq t \leq 1$, $dist(S,P) = \|S-P\|$

$= \|S - [(1-t)S + tT]\| = \|S + tS - S - tT\|$

$= \|t(S-T)\| = t\|S-T\|$;

$dist(P,T) = \|P-T\| = \|(1-t)S + tT - T\|$

$= \|S - tS + tT - T\| = \|S(1-t) - T(1-t)\|$

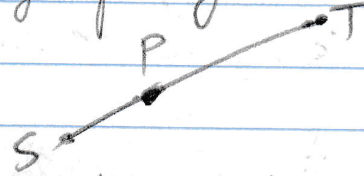
$= \|(1-t)(S-T)\| = (1-t)\|S-T\|$

Hence, $dist(P,S) = r[dist(P,T)]$ if and only if $t\|S-T\| = r(1-t)\|S-T\|$

$\longleftrightarrow t = r(1-t) \therefore t = \frac{r}{1+r}$ and $(1-t) = \frac{1}{1+r}$

It's amazing that exercise 18 can be answered simply by finding the ratio

$$\frac{\text{dist}(P,S)}{\text{dist}(P,T)}$$



The reason I am taking three pages to explain is because I wanted to **show** why $P = \frac{2}{3}S + \frac{1}{3}T$ and not the other way around.

I showed

$$\text{dist}(P,S) = r\left[\text{dist}(P,T)\right] \text{ if and only if}$$

$$t\|S-T\| = r(1-t)\|S-T\|$$

$$\iff \quad t = r(1-t)$$

Using this equation, we find values of t and $(1-t)$ in terms of r.

$$t = r - rt \iff t + rt = r$$

$$\iff t(1+r) = r \iff \boxed{t = \frac{r}{1+r}}$$

So $1 - t = 1 - \dfrac{r}{1+r}$

$$\boxed{1 - t = \frac{1+r-r}{1+r} = \frac{1}{1+r}}$$

So, we use this information to locate the point P of a line segment $\overline{ST}$ that partitions $\overline{ST}$ into two segments whose lengths are in ratio r.

Thus $P = \frac{1}{1+r} S + \frac{r}{1+r} T$

While this formula was not necessary for solving exercise 18, I thought it best to draw a simple graphical representation and plug ratio r into this formula to show that the $(1-t) = \frac{1}{1+r}$ part gets multiplied by point S and the $t = \frac{r}{1+r}$ part gets multiplied by the point T if we have to find point P between S and T.

I am sure that I will be using this formula in the remaining exercises, and I wanted to explain where it comes from. [see #23 to #28]

One may wonder "why bother?"
Or "why should I care to understand the formula?"

see whybother.freeboards.org [look up "Holden"]

{19, 22} Find k such that P lies between S and T.

(19) $P(1, k)$; $S(-2, -3)$; $T(5, 4)$

Note: In exercises 19 to 22 the method is to find the t for which
$P \in \overline{ST} = \{ S + t(T-S) \}$, $0 \leq t \leq 1$ and then solve for k.

Since P is between S and T,
$0 < t < 1$.

An alternating method would be to use the distance formula, which involves radicals.

So, for exercise #19:

$P \in \overline{ST} = \{S + t(T-S)\}$

so $P = S + t(T-S)$

$(1, k) = (-2, -3) + t[(5,4) - (-2, -3)]$

$= (-2, -3) + t(7, 7) = (-2 + 7t, -3 + 7t)$

$1 = -2 + 7t$ and $k = -3 + 7t$

$7t = 3$ and $k = -3 + 7\left(\frac{3}{7}\right) = 0$

$t = \frac{3}{7}$

(20) $P\left(k, \frac{4}{5}\right)$; $S(-1,5)$; $T(4,-2)$

$P \in \overline{ST} = \{ S + t(T-S) \}$

so $P = S + t(T-S)$

$$\left(k, 4/5\right) = (-1,5) + t\left[(4,-2)-(-1,5)\right]$$

$$= (-1,5) + t(5,-7) = (-1+5t, 5-7t)$$

$k = -1 + 5t$ and $\frac{4}{5} = 5 - 7t$

$$7t = 5 - \frac{4}{5} = \frac{25-4}{5} = \frac{21}{5}$$

$$t = \frac{21}{35} = \frac{3}{5} \text{ so } k = -1 + 5\left(\frac{3}{5}\right) = 2$$

Note about using distance formula

A more "brute force" method would be to use the theorem that if P, S, and T are distinct points in the plane, then P is between S and T if and only if dist(S, P) + dist(P, T) = dist(S, T).

So, for #20 $\|S-P\| + \|P-T\| = \|S-T\|$

$$\left\|(-1,5)-\left(k,\tfrac{4}{5}\right)\right\| + \left\|\left(k,\tfrac{4}{5}\right)-(4,-2)\right\| = \|(-1,5)-(4,-2)\|$$

$$\left\|\left(-1-k, \tfrac{21}{5}\right)\right\| + \left\|\left(k-4, \tfrac{14}{5}\right)\right\| = \|(-5,7)\|$$

[and solve for k]

As you can see, this involves [MUCH MORE] DRUDGERY.

(21) $P(1,4)$; $S(-3,1)$; $T(5,k)$

We will use the more "elegant" method, that is, finding t for which
$P \in \overline{ST} = \{S + t(T-S)\}$,
$0 \le t \le 1$ and solve for k.

$$(1,4) = (-3,1) + t[(5,k) - (-3,1)]$$

$$(1,4) = (-3,1) + t(8, k-1)$$

$$(1,4) = (-3+8t,\ 1 + t(k-1))$$

$$1 = -3 + 8t \leftrightarrow 8t = 4 \Leftrightarrow t = \frac{1}{2}$$

$$4 = 1 + t \cdot k - t = 1 + \frac{k}{2} - \frac{1}{2}$$

$$\frac{k}{2} = 4 - \frac{1}{2} = \frac{7}{2} \quad \therefore k = 7$$

(22) $P(1,-1);\ S(k,0);\ T(5,-3)$

$$P \in \overline{ST} = \{S + t(T-S)\}$$

$$P = S + t(T-S)$$

$$(1,-1) = (k,0) + t\left[(5,-3) - (k,0)\right]$$

$$(1,-1) = (k,0) + t\,(5-k,-3)$$

$$1 = k + t(5-k) = k + 5t - tk$$

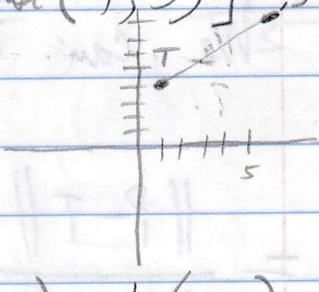
$$1 = k(1-t) + 5t \leftrightarrow 1-5t = k(1-t)$$

$$k = \frac{1-5t}{1-t}$$

$$-1 = -3t \leftrightarrow t = \frac{1}{3}$$

$$k = \frac{1-5\left(\frac{1}{3}\right)}{1-\frac{1}{3}} = \frac{1-\frac{5}{3}}{1-\frac{1}{3}} = \frac{-\frac{2}{3}}{\frac{2}{3}} = -1$$

$\{23, 28\}$ In each exercise find a point P on $\overline{TS}$ such that $dist(T,P) = r\,[dist(P,S)]$



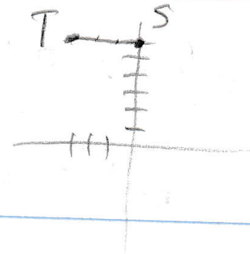
(23) $T(1,4);\ S(5,8);\ r = \frac{1}{2}$

Recall the formula $P = \frac{1}{1+r}T + \frac{r}{1+r}S$

$$P = \frac{1}{1+\frac{1}{2}}(1,4) + \frac{\frac{1}{2}}{1+\frac{1}{2}}(5,8) = \frac{2}{3}(1,4) + \frac{1}{3}(5,8)$$

$$P = \left(\frac{2}{3}, \frac{8}{3}\right) + \left(\frac{5}{3}, \frac{8}{3}\right) = \left(\frac{7}{3}, \frac{16}{3}\right)$$

(24) $T(-3,6)$; $S(0,6)$; $r = \frac{2}{3}$



Can you guess? $P = (-2,6)$?

$$P = \frac{1}{1+r} T + \frac{r}{1+r} S$$

$$P = \frac{1}{1+\frac{2}{3}}(-3,6) + \frac{\frac{2}{3}}{1+\frac{2}{3}}(0,6)$$

$$= \frac{3}{5}(-3,6) + \frac{2}{5}(0,6)$$

$$= \left(-\frac{9}{5}, \frac{18}{5}\right) + \left(0, \frac{12}{5}\right) = \left(-\frac{9}{5}, 6\right)$$

Do you see how untrustworthy our intuition is?

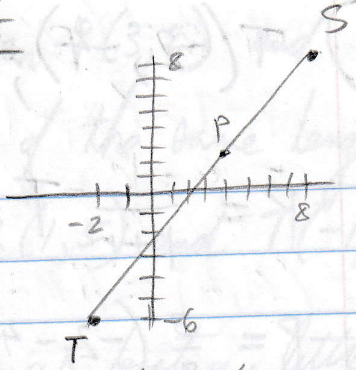
We cannot just think P will be $\frac{2}{3}$ between T and S. The ratio is of the distance between P and T and the distance between P and S.

We can check this: $\dfrac{\text{dist}(P,T)}{\text{dist}(P,S)}$

$$= \frac{\|P-T\|}{\|P-S\|} = \frac{\|(-\frac{9}{5},6)-(-3,6)\|}{\|(-\frac{9}{5},6)-(0,6)\|} = \frac{\|(\frac{6}{5},0)\|}{\|(-\frac{9}{5},0)\|}$$

$$= \frac{\sqrt{\frac{36}{25}}}{\sqrt{\frac{81}{25}}} = \frac{\frac{6}{5}}{\frac{9}{5}} = \frac{6}{9} = \frac{2}{3} = r$$

(25) $T(-2,-6)$; $S(8,8)$; $r=\frac{3}{2}$



$$P=\frac{1}{1+r}T+\frac{r}{1+r}S$$

$$P=\frac{1}{1+\frac{3}{2}}(-2,-6)+\frac{\frac{3}{2}}{1+\frac{3}{2}}(8,8)$$

$$P=\frac{2}{5}(-2,-6)+\frac{3}{5}(8,8)=\left(\frac{-4}{5},\frac{-12}{5}\right)+\left(\frac{24}{5},\frac{24}{5}\right)$$

$$P=\left(4,\frac{12}{5}\right)$$

Do you want to check $\dfrac{dist(P,T)}{dist(P,S)}=\dfrac{3}{2}$?

$$\frac{dist(P,T)}{dist(P,S)}=\frac{\|P-T\|}{\|P-S\|}=\frac{\|(4,\frac{12}{5})-(-2,-6)\|}{\|(4,\frac{12}{5})-(8,8)\|}=\frac{\|(6,\frac{42}{5})\|}{\|(-4,\frac{-28}{5})\|}$$

[YUCK! $\Rightarrow$ "drudgery", "toil", "agony"]

$$\frac{dist(P,T)}{dist(P,S)}=\frac{\sqrt{36+\frac{1764}{25}}}{\sqrt{16+\frac{784}{25}}}=\frac{\sqrt{\frac{2664}{25}}}{\sqrt{\frac{1184}{25}}}=\frac{\frac{6\sqrt{74}}{5}}{\frac{4\sqrt{74}}{5}}$$

$$=\frac{6}{4}=\frac{3}{2}=r$$

(26) $T(-2,-4)$; $S(-3,6)$; $r=\frac{3}{5}$

$$P=\frac{1}{1+r}T+\frac{r}{1+r}S=\frac{1}{1+\frac{3}{5}}T+\frac{\frac{3}{5}}{1+\frac{3}{5}}S$$

$$P=\frac{5}{8}(-2,-4)+\frac{3}{8}(-3,6)$$

$$\boxed{\begin{array}{l}\text{notice that } P=(1-t)T+tS \text{ where } t=\frac{3}{8}\\ (1-t)=\frac{1}{1+r} \text{ and } t=\frac{r}{1+r}\end{array}}$$

$$P=\left(\frac{-10}{8},\frac{-20}{8}\right)+\left(-\frac{9}{8},\frac{18}{8}\right)=\left(\frac{-19}{8},-\frac{1}{4}\right)$$

(27) $T(1,4)$; $S(-2,3)$; $r=2$

$$P=\frac{1}{1+r}T+\frac{r}{1+r}S=\frac{1}{3}T+\frac{2}{3}S$$

$$P=\frac{1}{3}(1,4)+\frac{2}{3}(-2,3)=\left(\frac{1}{3},\frac{4}{3}\right)+\left(\frac{-4}{3},2\right)$$

$$P=\left(-1,\frac{10}{3}\right)$$

(28) $T(5,2)$; $S(-2,8)$; $r=\frac{4}{3}$

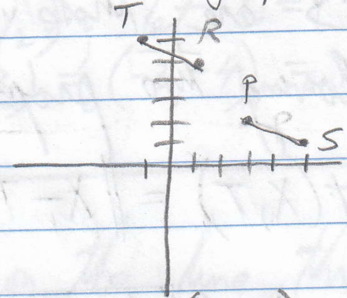
$$P=\frac{1}{1+r}T+\frac{r}{1+r}S=\frac{1}{1+\frac{4}{3}}T+\frac{\frac{4}{3}}{1+\frac{4}{3}}S$$

$$=\frac{3}{7}(5,2)+\frac{4}{7}(-2,8)=\left(\frac{15}{7},\frac{6}{7}\right)+\left(\frac{-8}{7},\frac{32}{7}\right)$$

$$=\left(1,\frac{38}{7}\right)$$

(29) Show that the segment between $P(3,2)$ and $S(5,1)$ is parallel to and of the same length as the segment between $R(1,5)$ and $T(-1,6)$.

Shall we try to relax and have at least a little fun? Why not first draw a graphical representation?



To show $\overline{PS} \parallel \overline{RT}$, we compare direction vectors:

$P-S = (3,2)-(5,1) = (-2,1)$

$R-T = (1,5)-(-1,6) = (2,-1) = -1\,(-2,1)$

$\therefore\ (P-S) \parallel (R-T)$ and $\overline{PS} \parallel \overline{RT}$

length of $\overline{PS} = \|P-S\| = \|(-2,1)\| = \sqrt{(-2)^2+1^2} = \sqrt{5}$

length of $\overline{RT} = \|R-T\| = \|(2,-1)\| = \sqrt{2^2+(-1)^2} = \sqrt{5}$

(30) Given $T(-1,4)$. Find two distinct points R and S such that $\overline{RT}$ and $\overline{TS}$ are both parallel and equal in length to the segment joining $M(-1,-2)$ and $N(2,2)$.

$M-N = (-1,-2)-(2,2) = (-3,-4)$

$R-T = (x_1,y_1)-(-1,4) = (x_1+1,\ y_1-4)$

$T-S = (-1,4)-(x_2,y_2) = (-1-x_2,\ 4-y_2)$

So $M-N=(-3,-4)$ $\boxed{T(-1,4)}$

$R-T=(x_1+1,\ y_1-4)$

$T-S=(-1-x_2,\ 4-y_2)$

$$dist(M,N)=\|M-N\|=\|(-3,-4)\|=\sqrt{(-3)^2+(-4)^2}$$
$$=\sqrt{9+16}=5$$

R and S are those points $X(x,y)$ with the following two properties

(1) $dist(X,T)=\|X-T\|=\|(x+1,\ y-4)\|$

$$=\sqrt{(x+1)^2+(y-4)^2}=dist(M,N)=5$$

(2) Since $\overline{XT}\parallel\overline{MN}$, $X-T=(x+1,\ y-4)$
$$=k(-3,-4)$$

$\therefore$ By (1) and (2) $\|X-T\|=\|k(-3,-4)\|=5$

$$\|(-3k,-4k)\|=5\leftrightarrow\sqrt{(-3k)^2+(-4k)^2}=5$$

$$\longleftrightarrow\sqrt{9k^2+16k^2}=\sqrt{25k^2}=5$$

$$5\sqrt{k^2}=5\longleftrightarrow\sqrt{k^2}=1\longleftrightarrow k=\pm1$$

Thus, from (2), $(x+1,\ y-4)=k(-3,-4)=\pm1(-3,-4)$

$k=1$ $x+1=-3$ and $y-4=-4\leftrightarrow x=-4,\ y=0$

$\therefore$ $(x, y) = (-4, 0)$ and $(x+1, y-4) = (3, 4)$,
for $k=1$ ⤴ for $k=-1$ $\to$ $(x, y) = (2, 8)$

We could say R is $(-4, 0)$ and S is $(2, 8)$
Check:
$R - T = (x_1 + 1, y_1 - 4)$. let $R = (-4, 0) = (x_1, y_1)$
then $R - T = (-4+1, 0-4) = (-3, -4) = 1(M-N)$
and $T - S = (-1 - x_2, 4 - y_2)$. let $S = (2, 8)$
then $T - S = (-1-2, 4-8) = (-3, -4) = 1(M-N)$

---

(31) Prove that $P(4,4)$ is on the line through the points $S(6,6)$ and $T(2,2)$ and is equidistant from them.

This means $r = 1$

So, $(4,4) = \frac{1}{1+1}(6,6) + \frac{1}{1+1}(2,2) = (3,3) + (1,1) = (4,4)$

$dist(S,T) = \|S - T\| = \|(4,4)\| = \sqrt{4^2 + 4^2} = \sqrt{32}$
$= 4\sqrt{2}$

$dist(S,P) = \|S - P\| = \|(2,2)\| = \sqrt{2^2 + 2^2} = \sqrt{8}$
$= 2\sqrt{2}$

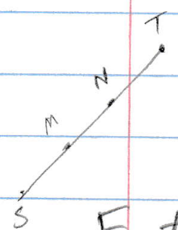
$dist(P,T) = \|P - T\| = \|(2,2)\| = 2\sqrt{2}$

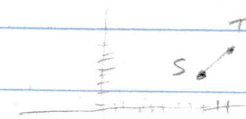
So, $dist(P,T) + dist(S,P) = dist(S,T)$
$\therefore$ P is between S and T and on the line through S and T.

and since $dist(S, P) = dist(P, T)$,

P is equidistant from T and S

(32) Given $S(8,4)$ and $T(10,8)$. Find two points M and N which divide $\overline{ST}$ into three segments of equal length.





First of all, we can rephrase this exercise to

"Find each of the points M and N which divide $\overline{ST}$ in a 1 to 2 ratio."

$$dist(S,N) = 2\left[dist(N,T)\right], \quad r_1 = 2$$

and $$dist(S,M) = \frac{1}{2}\left[dist(M,T)\right], \quad r_2 = \frac{1}{2}$$

So $$N = \frac{1}{1+2}S + \frac{2}{1+2}T = \frac{1}{3}S + \frac{2}{3}T$$

and $$M = \frac{1}{1+\frac{1}{2}}S + \frac{\frac{1}{2}}{1+\frac{1}{2}}T = \frac{2}{3}S + \frac{1}{3}T$$

$$N = \frac{1}{3}(8,4) + \frac{2}{3}(10,8) = \left(\frac{8}{3}, \frac{4}{3}\right) + \left(\frac{20}{3}, \frac{16}{3}\right) = \left(\frac{28}{3}, \frac{20}{3}\right)$$

$$M = \frac{2}{3}(8,4) + \frac{1}{3}(10,8) = \left(\frac{16}{3}, \frac{8}{3}\right) + \left(\frac{10}{3}, \frac{8}{3}\right) = \left(\frac{26}{3}, \frac{16}{3}\right)$$

(33) Prove: If P is the midpoint of $\overline{ST}$, then $P = \frac{1}{2}(S+T)$.

Since P is the midpoint of $\overline{ST}$, $dist(S,P) = dist(P,T)$ so $r=1$

$$P = \frac{1}{1+1}S + \frac{1}{1+1}T = \frac{1}{2}(S+T)$$

(34) Given: $P(2,3)$; $T(-4,-2)$, and $S(x,y)$.
If P is the midpoint of $\overline{ST}$, find x and y.

$$P = \frac{1}{2}S + \frac{1}{2}T$$

$$(2,3) = \frac{1}{2}(x,y) + \frac{1}{2}(-4,-2)$$

$$(2,3) = \left(\frac{x}{2} - 2, \frac{y}{2} - 1\right)$$

$$2 = \frac{x}{2} - 2 \quad \text{and} \quad 3 = \frac{y}{2} - 1$$

$$4 = \frac{x}{2} \leftrightarrow x = 8 \quad \text{and} \quad \frac{y}{2} = 4 \leftrightarrow y = 8$$

---

(35) Prove: If P, S, and T are distinct points in the plane, then T is between P and S if and only if $dist(P,T) + dist(T,S) = dist(P,S)$

T is between P and S if and only if
$T \in \overline{PS}$ and $T \neq P$ and $T \neq S$

This is equivalent to $T = (1-t)P + tS$, $0 < t < 1$.

(a) Assume T is between P and S:

$$\text{dist}(P,T) = \|T-P\| = \|(1-t)P + tS - P\|$$
$$= \|t(S-P)\| = t\|S-P\|;$$
$$\text{dist}(T,S) = \|S-T\| = \|S-(1-t)P - tS\|$$
$$= \|S(1-t) - P(1-t)\| = (1-t)\|S-P\|$$

$$\text{dist}(P,T) + \text{dist}(T,S) = t\|S-P\| + (1-t)\|S-P\|$$
$$= \|S-P\| = \text{dist}(P,S)$$

$$\boxed{* \text{ Note that } \text{dist}(A,B) = \|A-B\| = \|B-A\|}$$

(b) Assume (1) $\text{dist}(P,T) + \text{dist}(T,S) = \text{dist}(P,S)$

Suppose T does not lie between P and S.

Then for $0 < t < 1$,
$$T \neq (1-t)P + tS = P + t(S-P)$$

So, $T - P \neq t(S-P)$

② $\|T-P\| \neq \|t(S-P)\| = t\|S-P\|$ (for $0<t$).

From ①: $\|T-P\| + \|S-T\| = \|S-P\|$

$= t\|S-P\| + (1-t)\|S-P\|$.

From ②: $\|T-P\| + \|S-T\| \neq \|T-P\|$

$+ (1-t)\|S-P\|$,

$\|S-T\| \neq (1-t)\|S-P\| = \|(1-t)(S-P)\|$

(for $t<1$).

$\|S-T\| \neq \|S-(1-t)P - tS\| = \|S-T\|$

Contradiction!

$\therefore$ for $0<t<1$, $T = (1-t)P + tS$

and T lies between P and S

(36) Given $R(-7,-3)$, $S(x,y)$ and $P(3,5)$.

Determine x and y so that

$$\frac{\text{dist}(R,P)}{\text{dist}(P,S)} = \frac{2}{3}$$

$$\text{dist}(R,P) = \|R-P\| = \|(-10,-8)\|$$
$$= \sqrt{(-10)^2+(-8)^2} = \sqrt{164} = 2\sqrt{41}$$

$$\text{dist}(P,S) = \|P-S\| = \|(3-x,\ 5-y)\|$$
$$= \sqrt{(3-x)^2+(5-y)^2}.$$

Thus, $$\frac{2\sqrt{41}}{\sqrt{(3-x)^2+(5-y)^2}} = \frac{2}{3}$$

$$6\sqrt{41} = 2\sqrt{(3-x)^2+(5-y)^2}$$

$$(3\sqrt{41})^2 = (3-x)^2+(5-y)^2$$

$$\therefore\ (x,y) \in \{(x,y) : (x-3)^2+(y-5)^2 = 369\}$$

$\{(x,y)\}$ is a circle with center P and radius $\frac{3}{2}[\text{dist}(P,R)] = 3\sqrt{41}$

While this is the solution given in the solution key, I would have determined x and y explicitly.

$$\frac{dist(R,P)}{dist(P,S)} = \frac{2}{3} \text{ implies } P = \frac{1}{1+\frac{2}{3}} R + \frac{\frac{2}{3}}{1+\frac{2}{3}} S$$

$(3,5) = \frac{3}{5}(-7,-3) + \frac{2}{5}(x,y)$

$(3,5) = \left(\frac{-21}{5}, \frac{-9}{5}\right) + \left(\frac{2x}{5}, \frac{2y}{5}\right)$

$3 = -\frac{21}{5} + \frac{2x}{5}$ and $5 = \frac{-9}{5} + \frac{2y}{5}$

$15 = -21 + 2x$ $\qquad$ $25 = -9 + 2y$

$2x = 36$ $\qquad$ $2y = 34$

$x = 18$ $\qquad$ $y = 17$

This is how I initially answered Exercise 36. Does it agree with solution key?

$(3-18)^2 + (5-17)^2 = (-15)^2 + (-12)^2$

$= 225 + 144 = 369$

My answer gives the actual values of x and y.

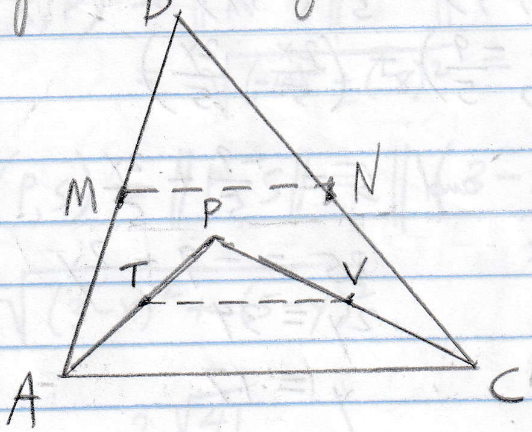
The solution key gives the value (x,y) in terms of the set $\{(x,y)\}$, which turns out to be a circle.

$369\}$

PLANE ANALYTIC GEOMETRY

5-6. PROVING GEOMETRIC THEOREMS

① In the figure shown, M, N, T, and V are respectively the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{AP}$, and $\overline{PC}$. Prove that $\overline{MN}$ is equal in length and parallel to $\overline{TV}$.



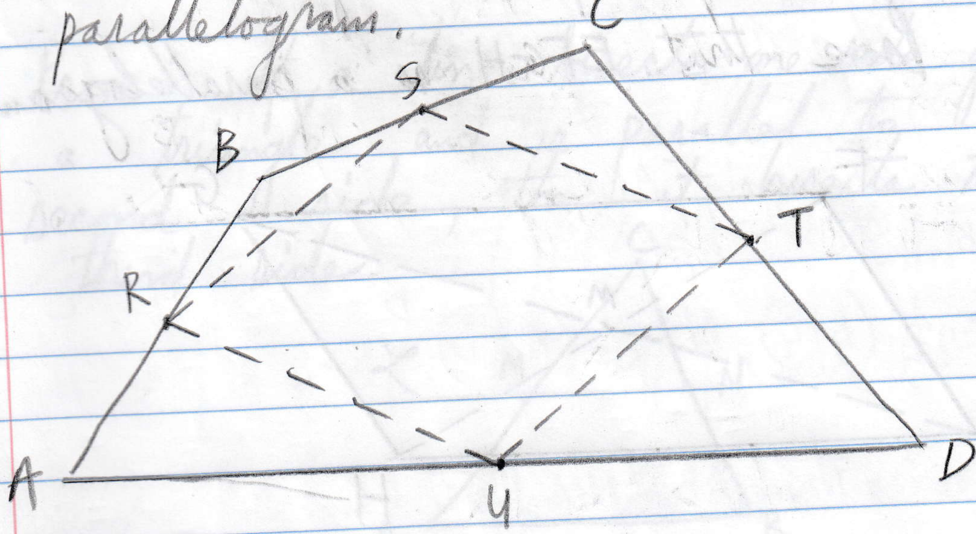
$M = \frac{1}{2}(A+B)$, $N = \frac{1}{2}(B+C)$, $T = \frac{1}{2}(A+P)$,
$V = \frac{1}{2}(P+C)$

$$M - N = \frac{1}{2}(A+B-B-C) = \frac{1}{2}(A-C)$$
$$T - V = \frac{1}{2}(A+P-P-C) = \frac{1}{2}(A-C)$$

$\therefore$ $M-N \parallel T-V$ and $\overline{MN} \parallel \overline{TV}$

Also $dist(M,N) = \|M-N\| = \|\frac{1}{2}(A-C)\|$
$= \|T-V\| = dist(T,V)$

(2) In quadrilateral ABCD, R, S, T, and U are respectively the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$. Prove that RSTU is a parallelogram.



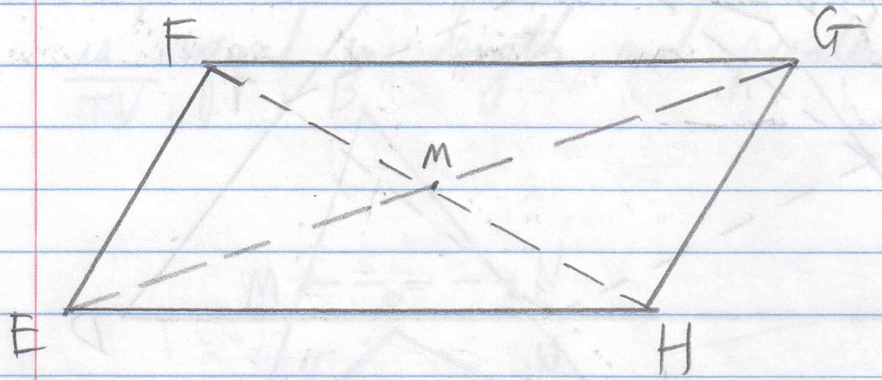
$R = \frac{1}{2}(A+B)$, $S = \frac{1}{2}(B+C)$, $T = \frac{1}{2}(C+D)$,
$U = \frac{1}{2}(A+D)$

Then $R-S = \frac{1}{2}(A+B-B-C) = \frac{1}{2}(A-C)$
and $T-U = \frac{1}{2}(C+D-A-D) = \frac{1}{2}(C-A)$

Since $A-C = -(C-A)$, $R-S \parallel T-U$
$\therefore \overline{RS} \parallel \overline{TU}$
Also $S-T = \frac{1}{2}(B+C-C-D) = \frac{1}{2}(B-D)$
and $U-R = \frac{1}{2}(A+D-A-B) = \frac{1}{2}(D-B)$
Since $B-D = -(D-B)$, $S-T \parallel U-R$
and $\overline{ST} \parallel \overline{UR}$
$\therefore$ quadrilateral RSTU is a parallelogram by definition.

(3) In the given quadrilateral EFGH, M is the midpoint of $\overline{EG}$ and also the midpoint of $\overline{FH}$.

Prove that EFGH is a parallelogram.



Since $M = \frac{1}{2}(E+G)$ and $M = \frac{1}{2}(F+H)$, $E+G = F+H$. Then $E-F = H-G$ and $\overline{EF} \parallel \overline{HG}$.

Also, $E-H = F-G$ and $\overline{EH} \parallel \overline{FG}$.

$\therefore$ EFGH is a parallelogram.

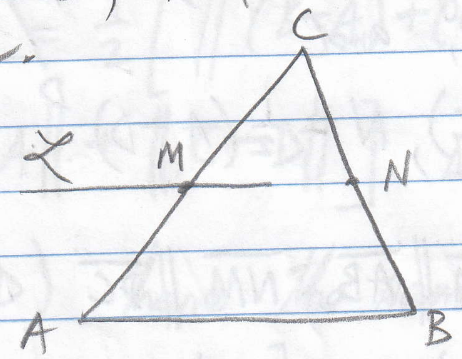
(4) In the figure for exercise 3 prove that if $\overline{FG}$ is parallel to $\overline{EH}$, then EFGH is a parallelogram.

(1) $F-G = t(E-H)$ for some $t \in \mathbb{R}$, $t > 0$

Also, (2) $\|F-G\| = \|E-H\|$.

For (1) and (2) to be true, $t=1$, and $F-G = (E-H)$, so that $\overline{HG} \parallel \overline{EF}$

(5) Prove: If a line bisects one side of a triangle and is parallel to a second side, then it bisects the third side.



In $\triangle ABC$, let $\mathcal{L} \parallel \overline{AB}$ and $M \in \mathcal{L}$ with $M = \frac{1}{2}(A+C)$.

Let $N = \frac{1}{2}(B+C)$. Prove $N \in \mathcal{L}$.

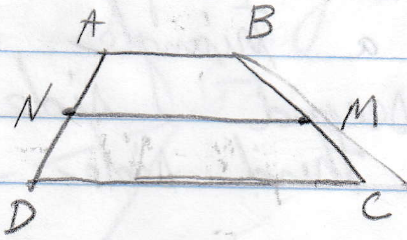
$\mathcal{L} \parallel \overline{AB}$ means $\mathcal{L} = \{M + t(B-A)\}$.

$N \in \mathcal{L}$ if and only if $N = M + t(B-A)$ for some t, that is, if and only if $\frac{1}{2}(B+C) = \frac{1}{2}(A+C) + t(B-A)$,

or $\frac{1}{2}(B-A) = t(B-A) \leftrightarrow t = \frac{1}{2}$

$\therefore$ for $t = \frac{1}{2}$, $N \in \mathcal{L}$ and bisects $\overline{CB}$.

⑥ Prove: The segment which joins the midpoints of the two nonparallel sides of a trapezoid (the median) is parallel to both bases and has length equal to half the sum of their lengths.



Given trapezoid $ABCD$, $\overline{AB} \parallel \overline{DC}$, and

$M = \frac{1}{2}(B+C)$, $N = \frac{1}{2}(A+D)$.

Prove: $\overline{NM} \parallel \overline{AB}$, $\overline{NM} \parallel \overline{DC}$,

$\text{dist}(N,M) = \frac{1}{2}\left[\text{dist}(A,B) + \text{dist}(D,C)\right]$,

$M-N = \frac{1}{2}(B+C-A-D) = \frac{1}{2}\left[(B-A)+(C-D)\right]$,

$B-A = t(C-D)$, $t > 0$, since the sides $\overline{AB}$ and $\overline{DC}$ are parallel and in the same direction.

$M-N = \frac{1}{2}\left[t(C-D)+(C-D)\right] = \frac{1}{2}(1+t)(C-D)$,

letting $s_1 = \frac{1}{2}(t+1)$, $M-N = s_1(C-D)$

$\therefore \overline{NM} \parallel \overline{DC}$;

$$(C-D) = \frac{1}{t}(B-A),$$

$$M-N = \frac{1}{2}\left(B-A+\frac{1}{t}(B-A)\right) = \frac{1}{2}\left(\frac{t+1}{t}\right)(B-A),$$

letting $S_2 = \frac{1}{2}\left(\frac{t+1}{t}\right)$, $M-N = S_2(B-A)$,

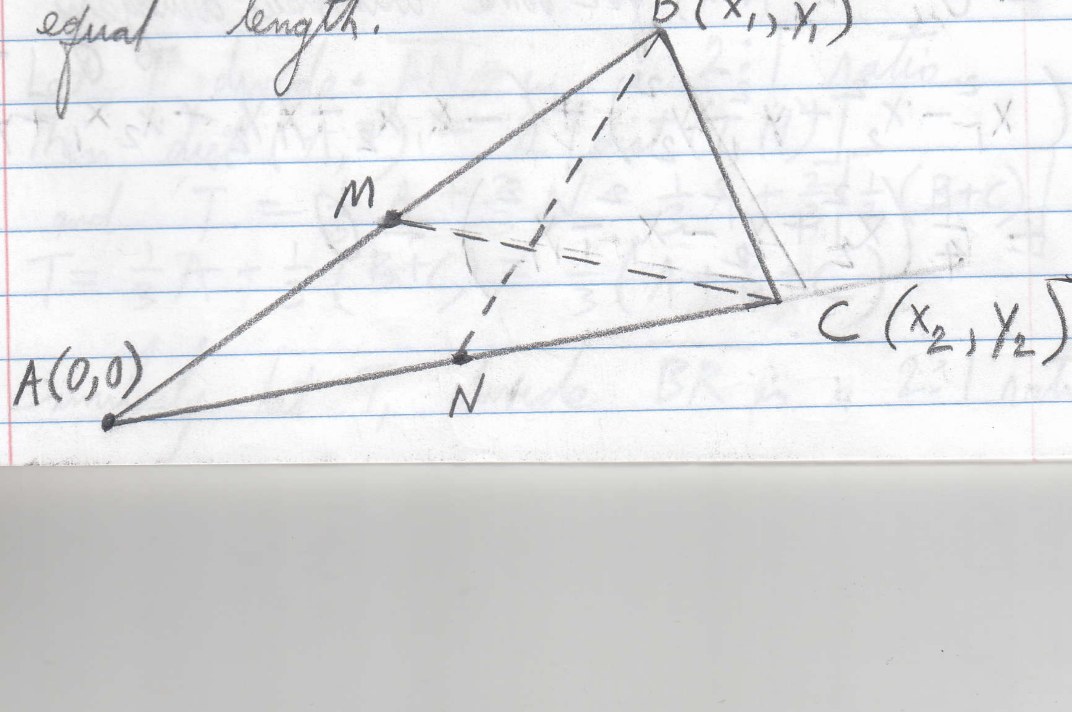
$\therefore \overline{NM} \parallel \overline{AB}$, $M-N = \frac{1}{2}\left[(B-A)+(C-D)\right]$

$$\therefore \|M-N\| = \frac{1}{2}\left[\|(B-A)+(C-D)\|\right]$$

$= \frac{1}{2}\left[\|B-A\| + \|C-D\|\right]$ since $(B-A)$ and $(C-D)$ have the same direction.

$$\therefore \text{dist}(N,M) = \frac{1}{2}\left[\text{dist}(A,B) + \text{dist}(D,C)\right].$$

7. In the given triangle, assume that $\overline{AB}$ and $\overline{AC}$ are segments of equal length. Prove that the medians to these sides, $\overline{CM}$ and $\overline{BN}$, are of equal length.



As midpoints, $M = \frac{1}{2}(A+B)$ and $N = \frac{1}{2}(A+C)$.

Since $A = (0,0)$, $M = \frac{B}{2} = \left(\frac{x_1}{2}, \frac{y_1}{2}\right)$ and $N = \frac{C}{2} = \left(\frac{x_2}{2}, \frac{y_2}{2}\right)$

(1) $\|B-N\| = \|C-M\|$ if and only if

$\|B-N\|^2 = \|C-M\|^2$ or $\|B-N\|^2 - \|C-M\|^2 = 0$.

This is true if and only if:

$$\left\|(x_1, y_1) - \left(\frac{x_2}{2}, \frac{y_2}{2}\right)\right\|^2 - \left\|(x_2, y_2) - \left(\frac{x_1}{2}, \frac{y_1}{2}\right)\right\|^2 = 0$$

$$\left[\left(x_1 - \frac{x_2}{2}\right)^2 + \left(y_1 - \frac{y_2}{2}\right)^2\right] - \left[\left(x_2 - \frac{x_1}{2}\right)^2 + \left(y_2 - \frac{y_1}{2}\right)^2\right] = 0$$

[Now, for some "arithmetic drudgery"]

$$(x_1^2 - x_2^2 + y_1^2 - y_2^2) + (-x_1 x_2 - y_1 y_2 + x_2 x_1 + y_2 y_1) + \frac{1}{4}(x_2^2 + y_2^2 - x_1^2 - y_1^2) = 0$$

(2) $\frac{3}{4}\left(x_1^2 - x_2^2 + y_1^2 - y_2^2\right) = 0$;

Since $\|B-A\| = \|C-A\|$, $\sqrt{x_1^2+y_1^2} = \sqrt{x_2^2+y_2^2}$,

$x_1^2 + y_1^2 = x_2^2 + y_2^2$

and $x_1^2 - x_2^2 + y_1^2 - y_2^2 = 0$

$\therefore$ (2) is true and thus (1) is true

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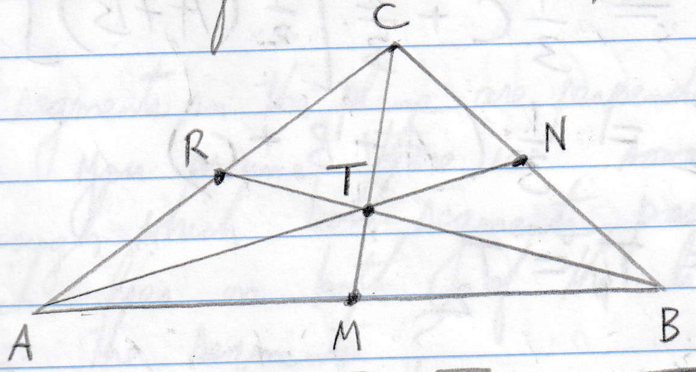
(8) Prove that the medians of any triangle meet in a point whose distance from each vertex is two-thirds the length of the median from that vertex.



The medians are $\overline{AN}$, $\overline{BR}$, and $\overline{CM}$ where

$N = \frac{1}{2}(B+C)$, $R = \frac{1}{2}(A+C)$, $M = \frac{1}{2}(A+B)$

Let T divide $\overline{AN}$ in a 2:1 ratio.

Then $\text{dist}(A,T) = 2\left[\text{dist}(T,N)\right]$

and $T = \frac{1}{3}A + \frac{2}{3}N = \frac{1}{3}A + \frac{2}{3}\left[\frac{1}{2}(B+C)\right]$

$T = \frac{1}{3}A + \frac{1}{3}(B+C) = \frac{1}{3}(A+B+C)$

Similarly, let T_1 divide $\overline{BR}$ in a 2:1 ratio.

then $\text{dist}(B, T_1) = 2\left[\text{dist}(T_1, R)\right]$

and $T_1 = \frac{1}{3}B + \frac{2}{3}R = \frac{1}{3}B + \frac{2}{3}\left[\frac{1}{2}(A+C)\right]$

so $T_1 = \frac{1}{3}(A+B+C)$

Let T_2 divide $\overline{CM}$ in a 2:1 ratio.
then $\text{dist}(C, T_2) = 2\left[\text{dist}(T_2, M)\right]$

and $T_2 = \frac{1}{3}C + \frac{2}{3}M$

$= \frac{1}{3}C + \frac{2}{3}\left[\frac{1}{2}(A+B)\right]$

so $T_2 = \frac{1}{3}(A+B+C)$

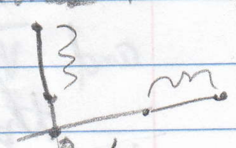
$\therefore$ $T = T_1 = T_2$

5-7 EQUATION OF A LINE

① Line $\mathcal{L}$ is perpendicular to line $\mathcal{N}$. The vector (r, s) is a normal vector to $\mathcal{L}$. Name a direction vector of $\mathcal{L}$, a direction vector of $\mathcal{N}$, and a normal vector to $\mathcal{N}$.

A direction vector of $\mathcal{L}$ is $(r, s)_p = (-s, r)$
A direction vector of $\mathcal{N}$ is (r, s)
A normal vector to $\mathcal{N}$ is $(-s, r)$

② If two segments in the plane are perpendicular, can you assume there is some point P
(a) through which both segments pass.
(b) which lies on both of the lines which contain the segments?



(a) No. the perpendicular segments may be contained in lines that intersect at a point not on the segments.

(b) Yes, if the segments are perpendicular, the lines containing them are perpendicular and must intersect.

In exercises 3 and 4 name three ordered pairs (x, y) for which the given statement is true.

(3) $(x,y)\cdot(2,3) = (6,1)\cdot(2,3)$

$2x + 3y = 12 + 3 = 15$

$\{(0,5), (6,1), (3,3)\}$

(4) $(x,y)\cdot(1,-2) = (4,3)\cdot(1,-2)$

$x - 2y = 4 - 6 = -2 \longleftrightarrow x - 2y = -2$

$\{(0,1), (2,2), (-2,0)\}$

In exercises 5-8 determine which one, or ones, of the given vectors $\vec{r}, \vec{s}, \vec{u}$, and $\vec{v}$ are normal vectors to the given line $\mathcal{L}$.

(5) $\mathcal{L} = \{(3,2) + t(4,1)\}$, $r = (-1,4)$

$\vec{s} = (-\frac{1}{2}, 2)$, $\vec{u} = (-3, 12)$, $\vec{v} = (-1, -4)$

$\vec{r}, \vec{s}, \vec{u}$; $(4,1)$ is a directional vector of $\mathcal{L}$.

$(4,1)\cdot(-1,4) = 0$; $(4,1)\cdot(-\frac{1}{2}, 2) = 0$;

$(4,1)\cdot(-3,12) = 0$; $(4,1)\cdot(-1,-4) = -8 \neq 0$

⑥ $\mathcal{L} = \{(-5,2) + q(-2,3)\}$; $r = (2,5)$,

$\vec{s} = (-3,2)$; $\vec{u} = (-3,-2)$, $\vec{v} = \left(-\frac{3}{2}, -1\right)$

$(-2,3)$ is a directional vector of $\mathcal{L}$.

$(-2,3)\cdot(2,5) = -4+15 = 11 \neq 0$

$(-2,3)\cdot(-3,2) = 6+6 = 12 \neq 0$

$(-2,3)\cdot(-3,-2) = 6-6 = 0$

$(-2,3)\cdot\left(-\frac{3}{2},-1\right) = 3-3 = 0$ $\therefore$ $\vec{u}$ and $\vec{v}$ are normal vectors to $\mathcal{L}$.

⑦ $\mathcal{L} = \{(2+3t, 1-t)\}$; $\vec{r} = (-1,3)$, $\vec{s} = (1,3)$, $\vec{u} = (-2,-6)$, $\vec{v} = (4,12)$

$\mathcal{L} = \{(2,1) + t(3,-1)\}$ so, a directional vector of $\mathcal{L}$ is $(3,-1)$.

$(3,-1)\cdot(-1,3) = -6 \neq 0$

$(3,-1)\cdot(1,3) = 3-3 = 0$

$(3,-1)\cdot(-2,-6) = -6+6 = 0$

$(3,-1)\cdot(4,12) = 12-12 = 0$

$\therefore$ $\vec{s}, \vec{u}, \vec{v}$ are normal vectors to $\mathcal{L}$.

(8) $\mathcal{L} = \{(5-2t, 1+t)\}$

$\vec{r} = (-1,-2)$, $\vec{s} = (-3,-6)$, $\vec{u} = (3,-6)$, $\vec{v} = (-1, \frac{1}{2})$

$\mathcal{L} = \{(5,1) + t(-2,1)\}$

so $(-2,1)$ is a directional vector of $\mathcal{L}$.

$(-2,1)\cdot(-1,-2) = 2-2 = 0$

$(-2,1)\cdot(-3,-6) = 6-6 = 0$

$(-2,1)\cdot(3,-6) = -6-6 = -12 \neq 0$

$(-2,1)\cdot(-1,\frac{1}{2}) = 2 + \frac{1}{2} = \frac{5}{2} \neq 0$

$\therefore$ $\vec{r}$ and $\vec{s}$ are normal vectors to $\mathcal{L}$.

In exercises 9 to 14 write an equation in the form $ax+by=c$ for the line through P having $\vec{n}$ as a normal vector.

(9) $P(5,2)$; $\vec{n} = (2,3)$

$X\cdot\vec{n} = P\cdot\vec{n} \longrightarrow (x,y)\cdot(2,3) = (5,2)\cdot(2,3)$

$2x + 3y = 16$

(10) $P(-2,3)$; $\vec{n} = (-1,4)$

$X\cdot\vec{n} = P\cdot\vec{n} \longrightarrow (x,y)\cdot(-1,4) = (-2,3)\cdot(-1,4)$

$-x + 4y = 14 \longleftrightarrow x - 4y = -14$

(11) $P(2,-3)$. $\vec{n} = (5,-1)$

$X\cdot\vec{n} = P\cdot\vec{n} \rightarrow (x,y)\cdot(5,-1) = (2,-3)\cdot(5,-1)$

$5x - y = 13$

(12) $P(-1,-1)$; $\vec{n} = (3,2)$

$X \cdot \vec{n} = P \cdot \vec{n} \rightarrow (x,y)\cdot(3,2) = (-1,-1)\cdot(3,2)$

$3x + 2y = -5$

(13) $P(4,1)$; $\vec{n} = (0,2)$

$X \cdot \vec{n} = P \cdot \vec{n} \rightarrow (x,y)\cdot(0,2) = (4,1)\cdot(0,2)$

$2y = 2 \leftrightarrow y = 1$

(14) $P(5,-1)$; $\vec{n} = (-3,0)$

$X \cdot \vec{n} = P \cdot \vec{n} \longrightarrow (x,y)\cdot(-3,0) = (5,-1)\cdot(-3,0)$

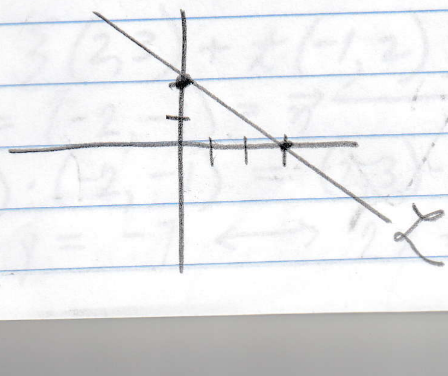
$-3x = -15 \leftrightarrow x = 5$

Name a normal vector and a direction vector for each given line $\mathcal{L}$. Then, make a representation of the given line in the coordinate plane.

(15) $\mathcal{L} = \{(x,y) : 2x + 3y = 6\}$

$2x + 3y = (x,y)\cdot(2,3)$

$\vec{n} = (2,3)$; $\vec{V} = \vec{n}_p = (-3,2)$

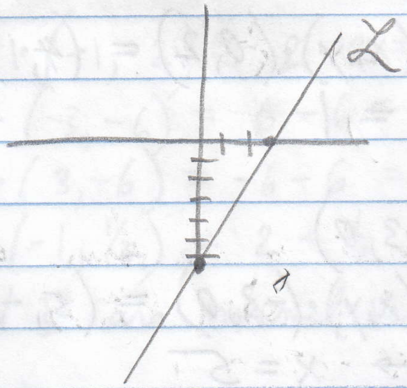


I just let $x=0$ to get y.
and let $y=0$ to get x.

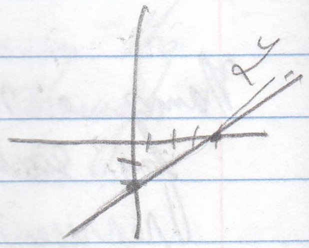
(16) $\mathcal{L} = \{(x,y) : 5x - 2y = 12\}$

$5x - 2y = (x,y) \cdot (5,-2)$

$\vec{n} = (5,-2)$; $\vec{v} = \vec{n}_p = (2,5)$



(17) $\mathcal{L} = \{(x,y) : x = 2y + 4\}$



$x - 2y = (x,y) \cdot (1,-2)$

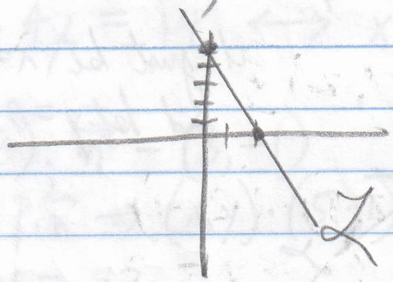
$\vec{n} = (1,-2)$; $\vec{v} = \vec{n}_p = (2,1)$

(18) $\mathcal{L} = \{(x,y) : 5x = 10 - 2y\}$

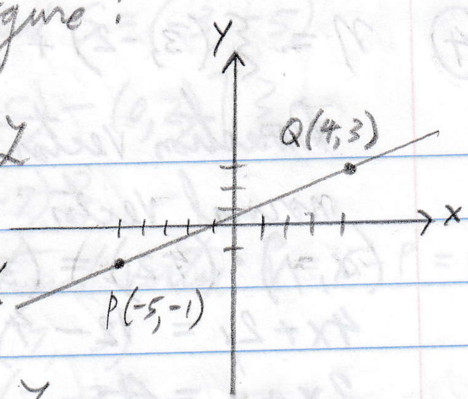
$= \{(x,y) : 5x + 2y = 10\}$

$5x + 2y = (x,y) \cdot (5,2)$

$\vec{n} = (5,2)$; $\vec{v} = (-2,5)$



Exercises 19 to 22 refer to the figure:



(19) Name a direction vector for $\mathcal{L}$

$\vec{v} = Q - P = (9,4)$

(20) Name a normal vector to $\mathcal{L}$.

$\vec{n} = (9,4)_p = (-4,9)$

(21) Write an equation for $\mathcal{L}$ in the form $ax + by = c$

$(x,y)\cdot(-4,9) = (4,3)\cdot(-4,9)$

$-4x + 9y = -16 + 27 = 11$

$4x - 9y = -11$

(22) Write an equation in the $ax+by=c$ form for a line through P perpendicular to $\mathcal{L}$.

$(x,y)\cdot(9,4) = (-5,-1)\cdot(9,4)$

$9x + 4y = -45 - 4 = -49$

Write the equation of the given line $\mathcal{M}$ in the form $ax + by = c$.

(23) $\mathcal{M} = \{(2,3) + t(-1,2)\}$

$\vec{v}_p = (-2,-1) = \vec{n}$

$(x,y)\cdot(-2,-1) = (2,3)\cdot(-2,-1)$

$-2x - y = -7 \longleftrightarrow 2x + y = 7$

(24) $\eta = \{(3,-2) + t(2,-4)\}$

Direction vector $\vec{v} = (2,-4)$

normal vector $\vec{n} = (4,2)$

$(x,y)\cdot(4,2) = (3,-2)\cdot(4,2)$

$4x + 2y = 12 - 4 = 8$

$2x + y = 4$

(25) $\eta = \{(-1,3) + q(0,2)\}$

direction vector $\vec{v} = (0,2)$

normal vector $\vec{n} = (-2,0)$

$(x,y)\cdot(-2,0) = (-1,3)\cdot(-2,0)$

$-2x = 2 \longleftrightarrow x = -1.$

(26) $\eta = \{(2,4+3q)\} = \{(2,4) + q(0,3)\}$

direction vector $\vec{v} = (0,3)$

normal vector $\vec{n} = (-3,0)$

$(x,y)\cdot(-3,0) = (2,4)\cdot(-3,0)$

$-3x = -6 \longleftrightarrow x = 2$

(27) $\eta = PS$, where $P = (-2,5)$ and $S = (1,8)$

$\eta = \{P + t(S-P)\} = \{(-2,5) + t(3,3)\}$

normal vector $\vec{n} = (-3,3)$

$(x,y)\cdot(-3,3) = (-2,5)\cdot(-3,3)$

$-3x + 3y = 6 + 15 = 21$

$x - y = -7$

(28) $n = \overline{PS}$, where $P = (0,2)$ and $S = (-3,-5)$

$n = \{ (0,2) + t[(-3,-5) - (0,2)] \}$

$= \{ (0,2) + t(-3,-7) \}$

so, line n has a directional vector $\vec{v} = S - P = (-3,-7)$
and a normal vector $\vec{n} = (7,-3)$

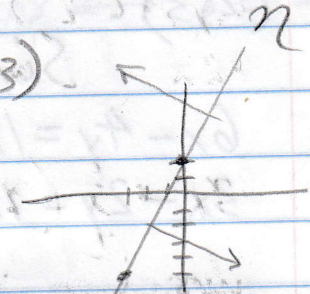
$(x,y)\cdot(7,-3) = (0,2)\cdot(7,-3)$

$7x - 3y = -6$

Notice that we get the same result if we choose directional vector as $P - S = (3,-7)$ and normal vector $\vec{n} = (7,-3)$

$(x,y)\cdot(7,-3) = (-3,-5)\cdot(7,-3)$

$7x - 3y = -21 + 15 = -6$



Which of the lines specified by the given equation are perpendicular to the line whose equation is $3x - 2y = 12$?

(29) $2x + 3y = 8$

$3x - 2y = (x,y)\cdot(3,-2)$

and $2x + 3y = (x,y)\cdot(2,3)$

$(3,-2)\cdot(2,3) = 6 - 6 = 0$ so $2x + 3y = 8$ is perpendicular to $3x - 2y = 12$.

$3x - 2y = 12$ has direction vector $(2,3)$.
A line will be perpendicular to $3x - 2y = 12$ when the dot product of the direction vectors is 0.

So, for $2x+3y=8$

Since $\vec{n}=(2,3)$, this line's direction vector is $\vec{n}_p=(-3,2)$.

$(2,3)\cdot(-3,2)=0$

(30) $6x-4y=11$. $6x-4y=(x,y)\cdot(6,-4)$ and $(6,-4)_p=(4,6)$, so a direction vector of $6x-4y=11$ is $(2,3)$.

A direction vector of $3x-2y=12$ is $(2,3)$.

Since $(2,3)\cdot(2,3)=4+9=13\neq 0$, $6x-4y=11$ is NOT perpendicular to $3x-2y=12$.

(31) $4x=10-6y \longleftrightarrow 4x+6y=10$

$4x+6y=(x,y)\cdot(4,6)$

$4x+6y=10$ has a direction vector $(-6,4)$

$(2,3)\cdot(-6,4)=-12+12=0$

So $4x=10-6y$ is perpendicular to $3x-2y=12$.

(32) $\frac{1}{3}x+\frac{1}{2}y=1 \longleftrightarrow 2x+3y=6$

$2x+3y=(x,y)\cdot(2,3)$ and $(2,3)_p=(-3,2)$

so $\frac{1}{2}x+\frac{1}{2}y=1$ has a direction vector $(-3,2)$.

$(2,3)\cdot(-3,2)=-6+6=0$ so $\perp$

For what value of k will $\mathcal{L}$ and $\mathcal{M}$ be perpendicular lines?

(33) $\mathcal{L} = \{(x,y) : 3x - 5y = 10\}$
and $\mathcal{M} = \{(x,y) : 6x + ky = 12\}$

$3x - 5y = (x,y)\cdot(3,-5)$ so a direction vector of $\mathcal{L}$ is $(3,-5)_p = (5,3)$
$6x + ky = (x,y)\cdot(6,k)$ so a direction vector of $\mathcal{M}$ is $(-k, 6)$.
$\mathcal{L} \perp \mathcal{M}$ if and only if $(5,3)\cdot(-k,6) = 0$
$-5k + 18 = 0 \longleftrightarrow 5k = 18 \longleftrightarrow k = \frac{18}{5}$

(34) $\mathcal{L} = \{(x,y) : 2x + 3y = 11\}$
$\mathcal{M} = \{(x,y) : 5x + ky = 10\}$

$2x + 3y = (x,y)\cdot(2,3)$, so a direction vector of $\mathcal{L}$ is $(-3,2)$.
$5x + ky = (x,y)\cdot(5,k)$, so a direction vector of $\mathcal{M}$ is $(-k, 5)$.
$\mathcal{L} \perp \mathcal{M}$ if and only if $(-3,2)\cdot(-k,5) = 0$
$3k + 10 = 0 \longleftrightarrow 3k = -10 \longleftrightarrow k = -\frac{10}{3}$

(35) $\mathcal{L} = \{(x,y): x - 2y = 4\}$ and

$\mathcal{N} = \{(x,y): kx + 3y = 8\}$

$x - 2y = (x,y)\cdot(1,-2)$, so a direction vector of $\mathcal{L}$ is $(2,1)$.

$kx + 3y = (x,y)\cdot(k,3)$, so a direction vector of $\mathcal{N}$ is $(-3,k)$.

$\mathcal{L} \perp \mathcal{N}$ if and only if $(2,1)\cdot(-3,k) = 0$:

$-6 + k = 0 \iff k = 6$

(36) $\mathcal{L} = \{(x,y): 2x + y = 5\}$ and

$\mathcal{N} = \{(x,y): kx - 2y = 7\}$

$2x + y = (x,y)\cdot(2,1)$, so a direction vector of $\mathcal{L}$ is $(-1,2)$.

$kx - 2y = (x,y)\cdot(k,-2)$, so a direction vector of $\mathcal{N}$ is $(2,k)$.

$\mathcal{L} \perp \mathcal{N}$ if and only if $(-1,2)\cdot(2,k) = 0$

$-2 + 2k = 0 \iff k = 1$

Determine an equation for the line $\mathcal{L}$ through point P perpendicular to line $\mathcal{N}$.

(37) $\mathcal{N} = \{(x,y): 3x + 2y = 6\}$; $P(4,-1)$

$3x + 2y = (x,y)\cdot(3,2)$

$\mathcal{L} = \{(4,-1) + t(3,2)\}$; $\vec{n} = (-2,3)$

$X \cdot \vec{n} = P \cdot \vec{n} \longrightarrow (x,y)\cdot(-2,3) = (4,-1)\cdot(-2,3)$

$-2x + 3y = -8 - 3 = -11$

$2x - 3y = 11$

Hence $\vec{n}_{\eta} = (3,2)$

$\vec{n}_{\mathcal{L}} = (-2,3)$

We let $\vec{n}_{\eta}$ denote normal vector of line η, and $\vec{n}_{\mathcal{L}}$ denote normal vector of line $\mathcal{L}$.

38 $\eta = \{(x,y) : 2x - 5y = 7\}$; $P(0,-3)$

$\vec{n}_{\eta} = (2,-5)$; $\vec{n}_{\mathcal{L}} = (5,2)$

$X \cdot \vec{n}_{\mathcal{L}} = P \cdot \vec{n}_{\mathcal{L}} \longrightarrow (x,y)\cdot(5,2) = (0,-3)\cdot(5,2)$

$5x + 2y = -6$

39 $\eta = \{(x,y) : 2x = 8\}$; $P(1,3)$

$2x = (x,y)\cdot(2,0)$ so $\vec{n}_{\eta} = (2,0)$

$\vec{n}_{\mathcal{L}} = (0,2)$

$X \cdot \vec{n}_{\mathcal{L}} = P \cdot \vec{n}_{\mathcal{L}} \longrightarrow (x,y)\cdot(0,2) = (1,3)\cdot(0,2)$

$2y = 6 \longleftrightarrow y = 3$

η is a vertical line $\therefore$ $\mathcal{L}$ is horizontal

(40) $\eta = \{(x,y): 3y = -6\}$; $P(-2,1)$

$$\vec{n}_\eta = (0,3);\quad \vec{n}_{\mathcal{L}} = (-3,0)$$

$$X \cdot \vec{n}_{\mathcal{L}} = P \cdot \vec{n}_{\mathcal{L}} \longrightarrow (x,y)\cdot(-3,0) = (-2,1)\cdot(-3,0)$$

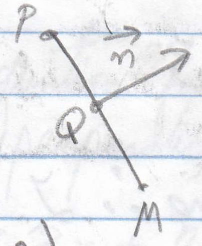
$$-3x = 6 \longleftrightarrow x = -2$$

η is a horizontal line $\therefore$ $\mathcal{L}$ must be vertical line $x = -2$

(41) Write a linear equation in the variables x and y for the line perpendicular to the segment joining $P(2,5)$ and $M(6,-1)$ at its midpoint.

A direction vector of $\overline{PM} = P - M = (-4,6)$

$(-4,6)$ is a normal $\vec{n}$ to the line that is perpendicular to $\overline{PM}$ at the midpoint Q.



$$Q = \tfrac{1}{2}(P+M) = \tfrac{1}{2}(8,4) = (4,2)$$

$$(x,y)\cdot(-4,6) = (4,2)\cdot(-4,6)$$

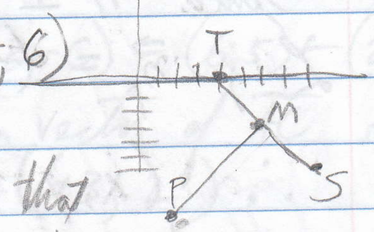
$$-4x + 6y = -16 + 12 = -4$$

$$4x - 6y = 4$$

$$2x - 3y = 2$$

(42) Determine k so that the point $P(1,k)$ lies on the line which is the perpendicular bisector of the segment joining $T(4,0)$ and $S(8,-6)$.

A direction vector of $\overline{TS} = T - S = (-4, 6)$



$(-4, 6)$ is a normal $\vec{n}$ to the line that is perpendicular to $\overline{TS}$ at the midpoint M.

$$M = \tfrac{1}{2}(T+S) = \tfrac{1}{2}(12,-6) = (6,-3)$$

$$(x,y)\cdot(-4,6) = (6,-3)\cdot(-4,6)$$

$$-4x + 6y = -24 - 18 = -42 \longleftrightarrow 2x - 3y = 21$$

$P(1,k) \in \{(x,y): 2x - 3y = 21\}$ if and only if

$$2(1) - 3k = 21$$

$$-3k = 19 \longleftrightarrow k = -\frac{19}{3}$$

Given the line $\mathcal{L} = \{(x,y): a\cdot x + by = c\}$ and the point $P(r,s)$ not on $\mathcal{L}$.

(43) Write a scalar equation for the line $\mathcal{N}$ through P and parallel to $\mathcal{L}$.

Since $\mathcal{N} \parallel \mathcal{L}$, normal $\vec{n}_{\mathcal{L}} = (a,b)$ is also normal $\vec{n}_{\mathcal{N}}$.

$$(x,y)\cdot(a,b) = (r,s)\cdot(a,b)$$

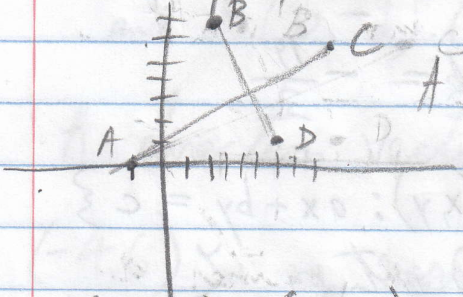
$$ax + by = a\cdot r + b\cdot s$$

(44) Write a scalar equation for the line $\mathcal{N}$ through P and perpendicular to $\mathcal{L}$.

Since $\mathcal{N} \perp \mathcal{L}$, the direction vector of $\mathcal{L}$, $\vec{n}_p = (-b, a)$, is normal to $\mathcal{N}$.

$$\therefore \quad (x, y) \cdot (-b, a) = (r, s) \cdot (-b, a)$$
$$-bx + ay = -br + as$$
$$xb - ay = br - as$$

(45) Given the four points $A(-1, 0)$, $B(2, 7)$, $C(8, 6)$, and $D(6, 1)$, show that $\overline{AC}$ is perpendicular to $\overline{BD}$.



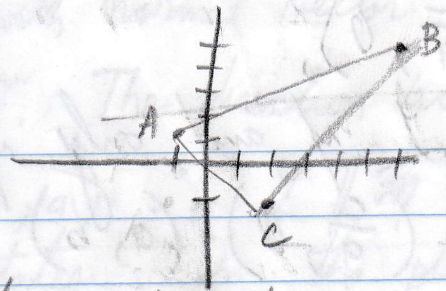
A direction vector of $\overline{AC}$ is $A - C$
$= (-9, -6)$

A direction vector of $\overline{BD}$ is $B - D$
$= (-4, 6)$

$$(A - C) \cdot (B - D) = (-9, -6) \cdot (-4, 6) = 36 - 36 = 0$$

$$\therefore \overline{AC} \perp \overline{BD}$$

(46) The vertices of triangle ABC are $A(-1, 1)$, $B(6, 5)$, and $C(2, -1)$. Show that the triangle is a right triangle.



A direction vector of $\overline{BC}$ is $(B-C) = (6,5)-(2,-1)$ $= (4,6)$. A direction vector of $\overline{AC}$ is $A-C = (-1,1)-(2,-1) = (-3,2)$.

$$(4,6)\cdot(-3,2) = -12+12 = 0$$

$\therefore \overline{AC} \perp \overline{BC}$ and ABC is a right triangle.

(47) Prove: If $(a,b) \neq \vec{0}$, $ax+by=c$ is an equation of the line having (a,b) as a normal vector and containing $(0,\frac{c}{b})$ if $b \neq 0$, and $(\frac{c}{a},0)$ if $a \neq 0$.

Solution: If $(a,b) \neq \vec{0}$, then either $a \neq 0$ or $b \neq 0$.

(1) $b \neq 0$. Consider the line $\mathcal{L}$ having (a,b) as a normal vector and containing $T = (0,\frac{c}{b})$.

$X = (x,y) \in \mathcal{L}$ if and only if

$$(x,y)\cdot(a,b) = (0,\tfrac{c}{b})\cdot(a,b),$$

$$ax+by = c$$

(2) $a \neq 0$; $T = (\frac{c}{a}, 0)$,

$X = (x, y) \in \mathcal{L}$ if and only if

$(x, y) \cdot (a, b) = (\frac{c}{a}, 0) \cdot (a, b)$

$ax + by = c$

(48) Prove: If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are both on the line $ax + by = c$, then $(y_1 - y_2, x_2 - x_1)$ is a normal vector to the line.

Solution: A direction vector of the line is

$Q - P = (x_2 - x_1, y_2 - y_1)$

$(Q - P)_p = (y_1 - y_2, x_2 - x_1)$

(49) Prove: There is exactly one line through a given point T having a given nonzero vector $\vec{n}$ as normal vector.

Solution: Through a given point T with given nonzero normal vector, there is at least one line, that is, the line with equation

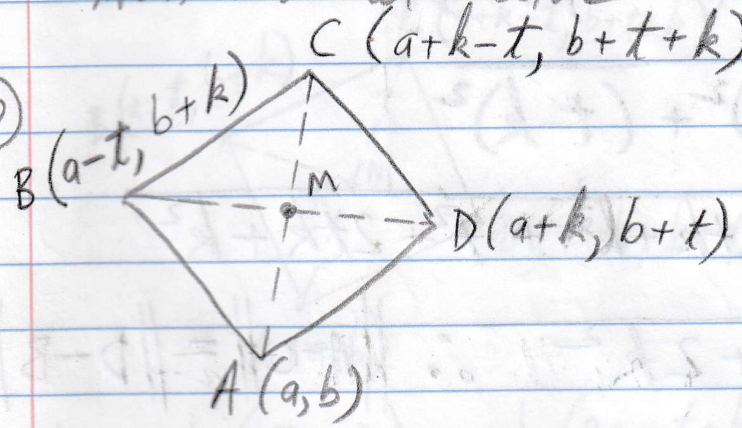
$X \cdot \vec{n} = T \cdot \vec{n}$.

Assume there are two such lines $\mathcal{L}$ and $\mathcal{M}$.

$\mathcal{L} \neq \mathcal{M}$ with normal vector $\vec{n} = (a, b)$ passing through T. The direction vector of $\mathcal{L}$ and $\mathcal{M}$ is $(-b, a)$ $\therefore$ $\mathcal{L} \parallel \mathcal{M}$, and by the theorem which states, "In the plane, lines having a point in common coincide if and only if they are parallel," since $\mathcal{L}$ and $\mathcal{M}$ have T in common, $\mathcal{L}$ and $\mathcal{M}$ must coincide. This is a contradiction.

$\therefore$ there is only one line through T with normal vector $\vec{n}$.

(50)



Prove that in the quadrilateral represented:

(a) the diagonals are perpendicular

$A - C = (t-k, -t-k)$

$D - B = (k+t, t-k)$

$(A-C)\cdot(D-B) = (t-k)(k+t) + (-t-k)(t-k)$

$= (t+k)(t-k) - (t+k)(t-k) = 0$

$\therefore \overline{AC} \perp \overline{BD}$

(b) the diagonals are equal in length.

$$\|A-C\| = \|(t-k, -(t+k))\|$$
$$= \sqrt{(t-k)^2 + (-(t+k))^2}$$
$$= \sqrt{t^2 - 2tk + k^2 + t^2 + 2tk + k^2}$$
$$= \sqrt{2t^2 + 2k^2}$$

$$\|D-B\| = \|(t+k, t-k)\|$$
$$= \sqrt{(t+k)^2 + (t-k)^2}$$
$$= \sqrt{t^2 + 2tk + k^2 + t^2 - 2tk + k^2}$$
$$= \sqrt{2t^2 + 2k^2} \quad \therefore \|A-C\| = \|D-B\|$$

(c) all pairs of adjacent sides are perpendicular.

$\overline{AD} \perp \overline{AB}$, $\overline{AB} \perp \overline{BC}$, $\overline{BC} \perp \overline{CD}$, $\overline{CD} \perp \overline{AD}$

$\overline{AD} \perp \overline{AB}$ if and only if $(A-D)\cdot(A-B) = 0$
$$(-k, -t)\cdot(t, -k) = -tk + tk = 0 \checkmark$$

$\overline{AB} \perp \overline{BC}$ if and only if $(A-B)\cdot(B-C) = 0$
$$(t, -k)\cdot(-k, -t) = -tk + tk = 0$$

$\overline{BC} \perp \overline{CD}$ if and only if $(B-C)\cdot(C-D)=0$

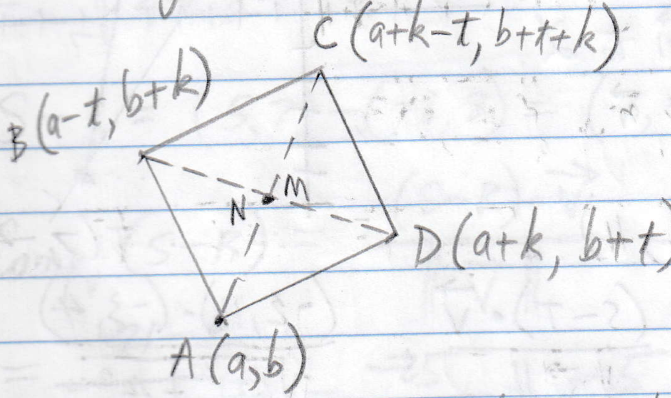
$(-k,-t)\cdot(-t,k) = tk - tk = 0$

$\overline{CD} \perp \overline{AD}$ if and only if $(C-D)\cdot(A-D)=0$

$(-t,k)\cdot(-k,-t) = tk - tk = 0$

$\therefore$ all pairs of consecutive sides are perpendicular

(d) the diagonals bisect each other



$$M = \frac{1}{2}(B+D) = \frac{1}{2}(2a-t+k,\ 2b+t+k)$$

$$= \left(a+\frac{k-t}{2},\ b+\frac{t+k}{2}\right)$$

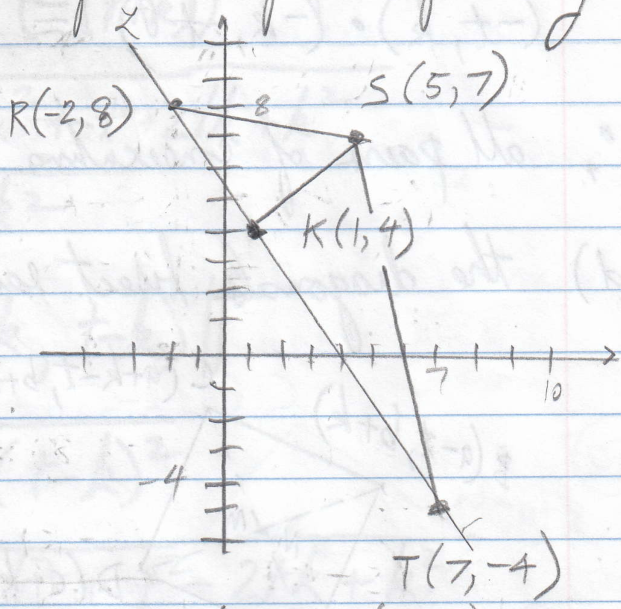
$$N = \frac{1}{2}(A+C) = \frac{1}{2}(2a+k-t,\ 2b+t+k)$$

$$= \left(a+\frac{k-t}{2},\ b+\frac{t+k}{2}\right)$$

$\therefore$ $M=N$ and each diagonal bisects the other.

5-8 DISTANCE BETWEEN A POINT AND A LINE

In the given figure name an ordered pair or a real number for each of the following



① A direction vector $\vec{v}$ of $\mathcal{L}$

$R - T = (-9, 12)$

$\vec{v} = (-3, 4)$

② a normal vector $\vec{n}$ to $\mathcal{L}$.

$\vec{v}_p = (-4, -3) = \vec{n}$

③ $S - T = (-2, 11)$

④ $\text{Comp}_v (S-T) = \dfrac{(S-T)\cdot\vec{v}}{\|\vec{v}\|} = \dfrac{(-2,11)\cdot(-3,4)}{\sqrt{(-3)^2+4^2}}$

$= \dfrac{6+44}{\sqrt{25}} = \dfrac{50}{5} = 10$

⑤ $\text{Comp}_{v_p} (S-T) = \dfrac{(S-T)\cdot\vec{v}_p}{\|\vec{v}_p\|} = \dfrac{(-2,11)\cdot(-4,-3)}{\sqrt{(-4)^2+(-3)^2}}$

$= \dfrac{8-33}{5} = \dfrac{-25}{5} = -5$

⑥ $\text{dist}(S, \mathcal{L}) = \dfrac{|(S-T)\cdot\vec{n}|}{\|\vec{n}\|} = \dfrac{|(-2,11)\cdot(-4,-3)|}{\sqrt{(-4)^2+(-3)^2}}$

$= \dfrac{|8-33|}{\sqrt{25}} = \dfrac{|-25|}{5} = 5$

note that $\text{Comp}_{v_p}(S-T) = \dfrac{(S-T)\cdot\vec{n}}{\|\vec{n}\|}$

$\text{dist}(S, \mathcal{L}) = \dfrac{|(S-T)\cdot\vec{n}|}{\|\vec{n}\|} = |\text{Comp}_{v_p}(S-T)|$

⑦ $\text{dist}(S, K) = \|S-K\|$

$= \|(5,7)-(1,4)\| = \|(4,3)\|$

$= \sqrt{4^2+3^2} = 5$

⑧ $S-R = (5,7)-(-2,8) = (7,-1)$

⑨ $\text{Comp}_{v}(S-R) = \dfrac{(S-R)\cdot\vec{v}}{\|\vec{v}\|} = \dfrac{(7,-1)\cdot(-3,4)}{\|(-3,4)\|}$

$= \dfrac{-21-4}{\sqrt{(-3)^2+4^2}} = \dfrac{-25}{5} = -5$

⑩ $\text{Comp}_{v_p}(S-R) = \dfrac{(S-R)\cdot\vec{v_p}}{\|\vec{v_p}\|} = \dfrac{(7,-1)\cdot(-4,-3)}{\|(-4,-3)\|}$

$= \dfrac{-28+3}{\sqrt{25}} = -5$

⑪ $\text{dist}(R, K) = \|(R-K)\| = \|(-2,8)-(1,4)\|$

$= \|(-3,4)\| = \sqrt{(-3)^2+4^2} = 5$

(12) $dist(S,R) = \|S-R\| = \|(5,7)-(-2,8)\|$

$= \|(7,-1)\| = \sqrt{7^2+1^2} = \sqrt{50} = 5\sqrt{2}$

For each exercise determine the distance between P and line $\mathcal{L}$ specified by the given equation.

(13) $\mathcal{L} = \{(4+t,\ 2-t)\}$; $P(6,2)$

$\mathcal{L} = \{(4,2) + t(1,-1)\}$, where $T = (4,2)$

$\vec{v} = (1,-1)$

$dist(P,\mathcal{L}) = \dfrac{|(P-T)\cdot\vec{n}|}{\|\vec{n}\|}$ $\quad \vec{n} = (1,1)$

$= \dfrac{|(2,0)\cdot(1,1)|}{\sqrt{1^2+1^2}} = \dfrac{2}{\sqrt{2}}\cdot\dfrac{\sqrt{2}}{\sqrt{2}} = \sqrt{2}$

(14) $\mathcal{L} = \{(3-2t,\ 1+t)\}$; $P(2,-1)$

$\mathcal{L} = \{(3,1) + t(-2,1)\}$

$T = (3,1)$; $\vec{v} = (-2,1)$; $\vec{n} = (-1,-2)$

$dist(P,\mathcal{L}) = \dfrac{|(P-T)\cdot\vec{n}|}{\|\vec{n}\|} = \dfrac{|(-1,-2)\cdot(-1,-2)|}{\sqrt{(-1)^2+(-2)^2}}$

$= \dfrac{|1+4|}{\sqrt{5}} = \dfrac{5}{\sqrt{5}}\cdot\dfrac{\sqrt{5}}{\sqrt{5}} = \dfrac{5\sqrt{5}}{5} = \sqrt{5}$

(15) $\mathcal{L} = \{(-1+q,\ 2-3q)\}$; $P(0,-1)$

$\mathcal{L} = \{(-1,2) + q(1,-3)\}$ so $T = (-1,2)$ and v

a direction vector of $\mathcal{L}$ is $\vec{v} = (1,-3)$ so a normal to $\mathcal{L}$ is $\vec{n} = (3,1)$

$$\text{dist}(P,\mathcal{L}) = \frac{|(P-T)\cdot\vec{n}|}{\|\vec{n}\|} = \frac{|(1,-3)\cdot(3,1)|}{\sqrt{3^2+1^2}} = \frac{|3-3|}{\sqrt{10}} = 0$$

Since PT and $\mathcal{L}$ have the same direction vector, they are parallel. Since PT and $\mathcal{L}$ have point T in common, PT and $\mathcal{L}$ coincide $\therefore$ $\text{dist}(P,\mathcal{L}) = 0$.

(16) $\mathcal{L} = \{(2-q,\ -3+q)\}$; $P(2,0)$

$\mathcal{L} = \{(2,-3) + q(-1,1)\}$ so $T = (2,-3)$;

a direction vector of $\mathcal{L}$ is $\vec{v} = (-1,1)$;

a normal to $\mathcal{L}$ is $\vec{n} = (-1,-1)$

$$\text{dist}(P,\mathcal{L}) = \frac{|(P-T)\cdot\vec{n}|}{\|\vec{n}\|} = \frac{|(0,3)\cdot(-1,-1)|}{\sqrt{(-1)^2+1^2}}$$

$$= \frac{|-3|}{\sqrt{2}} = \frac{3}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} = \frac{3}{2}\sqrt{2}$$

(17) $\mathcal{L} = \{(5,3) + r(1,-2)\}$; $P(0,0)$ $\quad \vec{v} = (1,-2)$, $\vec{n} = (2,1)$

$$\text{dist}(P,\mathcal{L}) = \frac{|(P-T)\cdot\vec{n}|}{\|\vec{n}\|} = \frac{|(-5,-3)\cdot(2,1)|}{\sqrt{2^2+1^2}} = \frac{|-13|}{\sqrt{5}}$$

$$= \frac{13}{\sqrt{5}}\frac{\sqrt{5}}{\sqrt{5}} = \frac{13}{5}\sqrt{5}$$

This agrees with $\text{dist}(P,\mathcal{L}) = \dfrac{c}{\sqrt{a^2+b^2}}$ $\quad a=2,\ b=1$, $c = (5,3)\cdot(2,1) = 13$

(18) $\mathcal{L} = \{(0,3) + a(-1,2)\}$; $P(-1,0)$

$$\text{dist}(P,\mathcal{L}) = \frac{|(-1,-3)\cdot(-2,-1)|}{\|(-2,-1)\|}$$

$$= \frac{|5|}{\sqrt{5}} = \frac{5}{\sqrt{5}}\frac{\sqrt{5}}{\sqrt{5}} = \sqrt{5}$$

(19) $3x - 4y = 10$; $P(2,1)$

$3x-4y = (x,y)\cdot(3,-4)$ so $\vec{n} = (3,-4)$

$\vec{n} = (a,b)$ so $a=3$, $b=-4$, $c=10$

$x^* = 2$, $y^* = 1$

$$\text{dist}(P,\mathcal{L}) = \frac{|ax^* + by^* - c|}{\sqrt{a^2+b^2}}$$

$$\text{dist}(P,\mathcal{L}) = \frac{|3\cdot2 - 4\cdot1 - 10|}{\sqrt{3^2+(-4)^2}} = \frac{|6-4-10|}{\sqrt{25}} = \frac{|-8|}{5}$$

$$= \frac{8}{5}$$

(20) $3x + 4y = 12$; $P(4,0)$ so $a=3$, $b=4$, $c=12$

$x^* = 4$, $y^* = 0$

$$\text{dist}(P,\mathcal{L}) = \frac{|3\cdot4 + 4\cdot0 - 12|}{\sqrt{3^2+4^2}} = \frac{|0|}{\sqrt{25}} = 0$$

(21) $5x - 12y = 10$; $P(3,1)$ so $a=5$, $b=-12$, $c=10$

$x^* = 3$, $y^* = 1$

$$\text{dist}(P,\mathcal{L}) = \frac{|5\cdot3 - 12\cdot1 - 10|}{\sqrt{5^2+(-12)^2}} = \frac{|-7|}{\sqrt{169}} = \frac{7}{13}$$

(22) $3x - y = 7$; $P(-1,2)$ so $a=3$, $b=-1$, $c=7$

$x^* = -1$, $y^* = 2$

$$\text{dist}(P,\mathcal{L}) = \frac{|3(-1) - 1\cdot 2 - 7|}{\sqrt{3^2+(-1)^2}} = \frac{|-3-2-7|}{\sqrt{10}} = \frac{|-12|}{\sqrt{10}}$$

$$= \frac{12\sqrt{10}}{10} = \frac{6}{5}\sqrt{10}$$

In each exercise the equations of two parallel lines are given. Find the distance between the two lines by selecting a point on one line and finding the distance between that point and the second line.

(23) $3x + 2y = 12$ and $3x + 2y = 16$

$(2,3)\cdot(3,2) = 12$

so $(2,3)$ is a point on the line. Let $P = (2,3)$.

let $x^* = 2$ and $y^* = 3$; $a=3$, $b=2$, $c=16$

$$\text{dist}(P,\mathcal{L}) = \frac{|(3\cdot 2 + 2\cdot 3 - 16)|}{\sqrt{3^2+2^2}} = \frac{|-4|}{\sqrt{13}} = \frac{4\sqrt{13}}{\sqrt{13}\sqrt{13}} = \frac{4}{13}\sqrt{13}$$

(24) $4x - 3y = 24$ and $8x - 6y = 36$

$4x - 3y = (x,y)\cdot(4,-3)$ so $\vec{n} = (a,b) = (4,-3)$

For $8x - 6y = 36$ when $x^* = 0$ and $y^* = -6$

For $4x - 3y = 24$, $a = 4$, $b = -3$, and $c = 24$

$$\frac{|4\cdot 0 - 3(-6) - 24|}{\sqrt{4^2+(-3)^2}} = \frac{|-6|}{\sqrt{25}} = \frac{6}{5}$$

(25) $\{\underset{T}{(2,1)} + r\underset{\vec{v}}{(3,2)}\}$ and $\{\underset{P}{(3,6)} + t\underset{\vec{v}}{(3,2)}\}$

$\vec{n} = (-2,3)$ so $a = -2$, $b = 3$

$(2,1)\cdot(-2,3) = -4+3 = -1 = c$ $\boxed{T\cdot\vec{n} = c}$

$P = (3,6)$ so $x^* = 3$, $y^* = 6$

$$\text{dist}(P,\mathcal{L}) = \frac{|(-2)(3)+(3)(6)-(-1)|}{\sqrt{(-2)^2+3^2}} = \frac{|13|}{\sqrt{13}}$$

$$= \frac{13}{\sqrt{13}}\cdot\frac{\sqrt{13}}{\sqrt{13}} = \sqrt{13}$$

We could have also chosen $T = (3,6)$, $P = (2,1)$

$\vec{n} = (-2,3)$. $a = -2$, $b = 3$, $x^* = 2$, $y^* = 1$,

$c = T\cdot\vec{n} = (3,6)\cdot(-2,3) = 12$

$$\text{dist}(P,\mathcal{L}) = \frac{|(-2)(2)+(3)(1)-12|}{\sqrt{(-2)^2+3^2}} = \frac{|-13|}{\sqrt{13}}$$

$$= \frac{13}{\sqrt{13}} = \sqrt{13}$$ Of course, the same result.

The lines are this distance from each other.

(26) $\{(4,1)+t(-3,4)\}$ and $\{(-2,1)+t(-6,8)\}$

For both lines $\vec{n} = (-8,-6)$ or $(4,3)$.

for $\{(-2,1)+t(-6,8)\}$, $(x,y)\cdot(4,3) = 4x+3y =$

$(-2,1)\cdot(4,3) = -8+3 = -5$, so $a = 4$, $b = 3$, $c = -5$,

$x^* = 4$, $y^* = 1$; $\dfrac{|4\cdot 4+3\cdot 1+5|}{\sqrt{4^2+3^2}} = \dfrac{24}{\sqrt{25}} = \dfrac{24}{5}$

Determine k so that P is at the given distance from line $\mathcal{L}$.

(27) $\mathcal{L} = \{(x,y) : 3x - 4y = k\}$; $P(2,3)$;

$\operatorname{dist}(P,\mathcal{L}) = \frac{6}{5}$

$3x - 4y = (x,y)\cdot(3,-4)$ so $\vec{n} = (3,-4)$

$a = 3,\ b = -4,\ c = k,\ x^* = 2,\ y^* = 3$

$$\operatorname{dist}(P,\mathcal{L}) = \frac{6}{5} = \frac{|3\cdot 2 - 4\cdot 3 - k|}{\sqrt{3^2+(-4)^2}} = \frac{|-6-k|}{\sqrt{25}}$$

$$\frac{6}{5} = \frac{|-(k+6)|}{5} \longleftrightarrow |-(k+6)| = 6$$

THINK! $-(k+6) = 6$ or $-(k+6) = -6$

so $k+6 = -6$ or $k+6 = 6$

$k = -12$ or $k = 0$

(28) $\mathcal{L} = \{(x,y) : 2x + 3y = k\}$; $P(5,3)$; $\operatorname{dist}(P,\mathcal{L}) = \sqrt{13}$

$a = 2,\ b = 3,\ c = k,\ x^* = 5,\ y^* = 3$

$$\operatorname{dist}(P,\mathcal{L}) = \frac{|2\cdot 5 + 3\cdot 3 - k|}{\sqrt{2^2+3^2}} = \sqrt{13}$$

$$\frac{|19-k|}{\sqrt{13}} = \sqrt{13} \longleftrightarrow |19-k| = 13$$

so $19-k = 13$ or $19-k = -13$

$k = 6$ or $k = 32$

(29) $\mathcal{L} = \{(4,3) + t(2,-1)\}$; $P(3k,k)$; $dist(P,\mathcal{L}) = 2\sqrt{5}$

A direction vector of $\mathcal{L}$ is $\vec{v} = (2,-1)$

a normal to $\mathcal{L}$ is $\vec{n} = (1,2)$

$(x,y)\cdot(1,2) = (4,3)\cdot(1,2)$

$x + 2y = 10$ so $a=1$, $b=2$, $c=10$

$x^* = 3k$, $y^* = k$.

$$dist(P,\mathcal{L}) = 2\sqrt{5} = \frac{|3k+2k-10|}{\sqrt{1^2+2^2}} = \frac{|5k-10|}{\sqrt{5}}$$

$$\frac{|5k-10|}{\sqrt{5}} = 2\sqrt{5} \longleftrightarrow |5k-10| = 10$$

$\therefore$ $5k-10 = 10$ or $5k-10 = -10$

$5k = 20$ $5k = 0$

$k = 4$ or $k = 0$

(30) $\mathcal{L} = \{(-2k,3k) + t(3,-4)\}$; $P(3,2)$;

$dist(P,\mathcal{L}) = 4$

a direction vector of $\mathcal{L}$ is $\vec{v} = (3,-4)$

so a normal to $\mathcal{L}$ is $\vec{n} = (4,3)$

$(x,y)\cdot(4,3) = (-2k,3k)\cdot(4,3)$

$4x + 3y = -8k + 9k = k$

so $a=4$, $b=3$, $c=k$

$x^* = 3$, $y^* = 2$

$$dist(P,\mathcal{L}) = 4 = \frac{|4\cdot 3 + 3\cdot 2 - k|}{\sqrt{4^2+3^2}}$$

$2\sqrt{5}$

$$\frac{|18-k|}{5} = 4 \leftrightarrow |18-k| = 20$$

$\therefore$ $18-k = 20$ or $18-k = -20$

$k = -2$ or $k = 38$

10|

(31) Given the line $\mathcal{L}$ with equation $12x + by + 36 = 0$. Determine b if $\mathcal{L}$ is at a distance of 3 from $P(0,0)$.

The distance between the line $\{(x,y) : ax + by = c\}$ and the origin is $\dfrac{|c|}{\sqrt{a^2+b^2}}$.

So, here, $a = 12$ and $c = -36$

$$dist(P, \mathcal{L}) = \frac{|-36|}{\sqrt{12^2+b^2}} = 3 \leftrightarrow \frac{36}{3} = \sqrt{144+b^2}$$

$$12 = \sqrt{144+b^2} \longleftrightarrow 144 = 144 + b^2$$

$b^2 = 0$ $\therefore$ $b = 0$

(32) Given the line η with equation $ax + 4y - 14 = 0$. Determine a if η is a distance of 1 from $P(1,3)$. So, $\eta = \{(x,y) : ax + 4y = 14\}$

$a = a$, $b = 4$, $c = 14$, $x^* = 1$, $y^* = 3$

$$dist(P, \eta) = 1 = \frac{|a + 4 \cdot 3 - 14|}{\sqrt{a^2+4^2}} = 1 \leftrightarrow |a-2| = \sqrt{a^2+16}$$

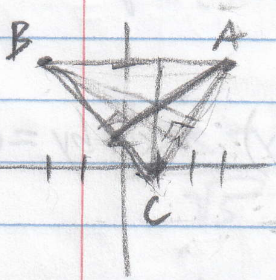
$|a-2| = \pm\sqrt{a^2+16}$

$$|a-2| = \sqrt{16+a^2} \quad \text{or} \quad |a-2| = -\sqrt{16+a^2}$$

$$a^2-4a+4 = 16+a^2 \iff -4a = 12 \iff a = -3$$

Find the length of the three altitudes of ΔABC

(33) $A(3,4)$; $B(-2,4)$; $C(1,0)$



h_X denotes length of altitude from vertex X.

$$h_A = \operatorname{dist}(A, \overline{BC}) = \frac{|(A-B)\cdot(B-C)_p|}{\|(B-C)_p\|}$$

$$h_B = \operatorname{dist}(B, \overline{AC}) = \frac{|(B-C)\cdot(C-A)_p|}{\|(C-A)_p\|}$$

$$h_C = \operatorname{dist}(C, \overline{AB}) = \frac{|(C-A)\cdot(A-B)_p|}{\|(A-B)_p\|}$$

$A-B = (3,4)-(-2,4) = (5,0)$

$B-C = (-2,4)-(1,0) = (-3,4)$

$C-A = (1,0)-(3,4) = (-2,-4)$

$(A-B)_p = (5,0)_p = (0,5)$

$(B-C)_p = (-3,4)_p = (-4,-3)$

$(C-A)_p = (-2,-4)_p = (4,-2)$

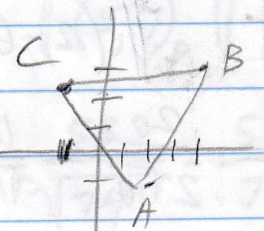
$$h_A = dist(A, \overline{BC}) = \frac{|(5,0)\cdot(-4,-3)|}{\|(-4,-3)\|} = \frac{|-20|}{\sqrt{25}} = 4$$

$$h_B = dist(B, \overline{AC}) = \frac{|(-3,4)\cdot(4,-2)|}{\|(4,-2)\|} = \frac{|-20|}{\sqrt{20}} = \frac{20}{2\sqrt{5}} = \frac{10}{\sqrt{5}}$$

$$= \frac{10\sqrt{5}}{5} = 2\sqrt{5}$$

$$h_C = dist(C, \overline{AB}) = \frac{|(-2,-4)\cdot(0,5)|}{\sqrt{25}} = \frac{|-20|}{5} = 4$$

(34) $A(2,-1);\ B(4,3);\ C(-1,2)$



$$h_A = dist(A, \overline{BC}) = \frac{|(A-B)\cdot(B-C)_p|}{\|(B-C)_p\|}$$

$$= \frac{|(-2,-4)\cdot(-1,5)|}{\|(-1,5)\|} = \frac{|-18|}{\sqrt{26}} = \frac{18}{\sqrt{26}}\cdot\frac{\sqrt{26}}{\sqrt{26}} = \frac{18\sqrt{26}}{26} = \frac{9}{13}\sqrt{26}$$

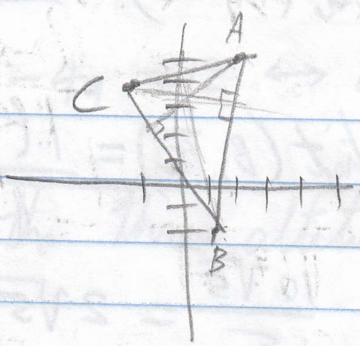
$$h_B = dist(B, \overline{AC}) = \frac{|(B-C)\cdot(C-A)_p|}{\|(C-A)_p\|} = \frac{|(5,1)\cdot(-3,-3)|}{\|(-3,-3)\|}$$

$$= \frac{|-18|}{\sqrt{18}} = \frac{18}{3\sqrt{2}} = \frac{6}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{6\sqrt{2}}{2} = 3\sqrt{2}$$

$$h_C = dist(C, \overline{AB}) = \frac{|(C-A)\cdot(A-B)_p|}{\|(A-B)_p\|} = \frac{|(-3,3)\cdot(4,-2)|}{\|(4,-2)\|}$$

$$= \frac{|-18|}{\sqrt{20}} = \frac{18}{2\sqrt{5}} = \frac{9}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}} = \frac{9}{5}\sqrt{5}$$

(35) $A(2,5);\ B(1,-2);\ C(-1,4)$



$$h_A = \operatorname{dist}(A, \overline{BC}) = \frac{|(A-B)\cdot(B-C)_p|}{\|(B-C)_p\|}$$

$$= \frac{|(1,7)\cdot(2,-6)_p|}{\|(2,-6)_p\|} = \frac{|(1,7)\cdot(6,2)|}{\|(6,2)\|} = \frac{20}{\sqrt{40}}$$

$$= \frac{20}{2\sqrt{10}} = \frac{10}{\sqrt{10}}\cdot\frac{\sqrt{10}}{\sqrt{10}} = \sqrt{10}$$

$$h_B = \operatorname{dist}(B, \overline{AC}) = \frac{|(B-C)\cdot(C-A)_p|}{\|(C-A)_p\|} = \frac{|(2,-6)\cdot(-3,-1)_p|}{\|(-3,-1)_p\|}$$

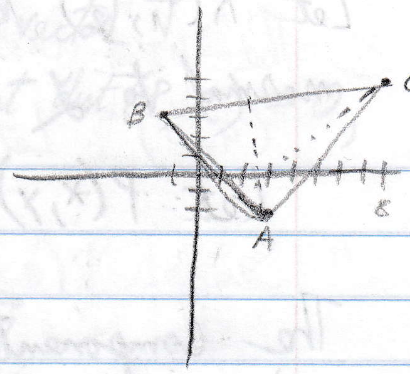
$$= \frac{|(2,-6)\cdot(1,-3)|}{\|(1,-3)\|} = \frac{20}{\sqrt{10}}\cdot\frac{\sqrt{10}}{\sqrt{10}} = 2\sqrt{10}$$

$$h_C = \operatorname{dist}(C, \overline{AB}) = \frac{|(C-A)\cdot(A-B)_p|}{\|(A-B)_p\|} = \frac{|(-3,-1)\cdot(1,7)_p|}{\|(1,7)_p\|}$$

$$= \frac{|(-3,-1)\cdot(-7,1)|}{\|(-7,1)\|} = \frac{|20|}{\sqrt{(-7)^2+1^2}} = \frac{20}{\sqrt{50}} = \frac{20}{5\sqrt{2}} = \frac{4}{\sqrt{2}}$$

$$= \frac{4\sqrt{2}}{2} = 2\sqrt{2}$$

(36) $A(3,-2)$; $B(-1,3)$; $C(8,5)$



$$h_A = \text{dist}(A, \overline{BC}) = \frac{|(A-B)\cdot(C-B)_p|}{\|(C-B)_p\|}$$

$$= \frac{|(4,-5)\cdot(9,2)_p|}{\|(9,2)_p\|} = \frac{|(4,-5)\cdot(-2,9)|}{\|(-2,9)\|}$$

$$= \frac{|-53|}{\sqrt{(-2)^2+9^2}} = \frac{53}{\sqrt{85}} = \frac{53\sqrt{85}}{85}$$

$$h_B = \text{dist}(B, \overline{AC}) = \frac{|(B-C)\cdot(C-A)_p|}{\|(C-A)_p\|} = \frac{|(-9,-2)\cdot(5,7)_p|}{\|(5,7)_p\|}$$

$$= \frac{|(-9,-2)\cdot(-7,5)|}{\|(-7,5)\|} = \frac{53}{\sqrt{(-7)^2+5^2}} = \frac{53}{\sqrt{74}} = \frac{53\sqrt{74}}{74}$$

$$h_C = \text{dist}(C, \overline{AB}) = \frac{|(C-A)\cdot(A-B)_p|}{\|(A-B)_p\|} = \frac{|(5,7)\cdot(4,-5)_p|}{\|(4,-5)_p\|}$$

$$= \frac{|(5,7)\cdot(5,4)|}{\|(5,4)\|} = \frac{|25+28|}{\sqrt{5^2+4^2}} = \frac{53}{\sqrt{41}} = \frac{53\sqrt{41}}{41}$$

(37) Using the Pythagorean Theorem, prove that $\text{dist}(S, \mathcal{L})$ is the length of the shortest line segment joining S to a point of $\mathcal{L}$.

Let $R(r_1, r_2)$ be the intersection of $\mathcal{L}$ and the unique normal to $\mathcal{L}$ through $S(s_1, s_2)$.

Let $P(x,y)$ be any point of $\mathcal{L}$, $P \neq R$.

The component of $S-P$ along the normal to $\mathcal{L}$ is $S-R$

$$\therefore \text{dist}(S, \mathcal{L}) = \frac{|S-P| \cdot \vec{n}}{\|\vec{n}\|} = \text{Comp}_n (S-P) = \|S-R\|$$

$$= \text{dist}(S,R)$$

$$\therefore \text{dist}(S,R) = \text{dist}(S, \mathcal{L}).$$

Since $\overline{SR} \perp \overline{RP}$, $\overline{SP}$ is the hypotenuse of the right triangle SRP.

By the Pythagorean Theorem,

$$\text{dist}(S,P)^2 = \text{dist}(S,R)^2 + \text{dist}(R,P)^2$$

$$\text{dist}(S,P) = \sqrt{\text{dist}(S,R)^2 + \text{dist}(R,P)^2} > \sqrt{\text{dist}(S,R)^2}$$

$= \text{dist}(S,R)$, $R \neq P$; since $\text{dist}(R,P)^2 > 0$

$\therefore$ for any $P \neq R$ on $\mathcal{L}$, $\text{dist}(S,P) > \text{dist}(S,R)$ and $\text{dist}(S,P) > \text{dist}(S,\mathcal{L})$ for any $P \neq R$.

(38) Given line $\mathcal{L}$ with a nonzero direction vector $\vec{v}$.
Prove that if P is a point such that $dist(P, \mathcal{L}) = 0$, then P lies on $\mathcal{L}$.

$\mathcal{L}$ has a nonzero direction vector $\vec{v}$
$\therefore$ a normal $\vec{n} \neq 0$
Let T be some point on $\mathcal{L}$

$$dist(P, \mathcal{L}) = \frac{|(P-T)\cdot\vec{n}|}{\|\vec{n}\|} = 0$$

$(P-T)\cdot\vec{n} = 0$, $T\cdot\vec{n} = P\cdot\vec{n}$ $\therefore$ P lies on $\mathcal{L}$.

A point X in the plane lies on $\mathcal{M}$ if and only if $X - T = 0$ or $X - T$ is a direction vector of $\mathcal{M}$. Therefore, $X \in \mathcal{M}$ if and only if

$$(X-T)\cdot\vec{n} = 0 \leftrightarrow X\cdot\vec{n} = T\cdot\vec{n}$$

Writing $X = (x, y)$ and $T\cdot\vec{n} = c$, we can express the equation of a line as $(x,y)\cdot(a,b) = c$
or $ax + by = c$,

(39) Write an equation of the line on which the point closest to the origin has the coordinates (a, b) where a and b are not both zero.

The point closest to the origin has coordinates (a, b). Let this be point P.

Since $P(a,b)$ nearer to the origin $Q(0,0)$ than any other point of $\mathcal{L}$, $\overline{PQ} \perp \mathcal{L}$,

$\text{dist}(P, Q) = \text{dist}(P, \mathcal{L})$

$(P-Q) = (a, b)$ is a normal vector to $\mathcal{L}$.

The desired equation is $(x, y)\cdot(a, b) = (a, b)\cdot(a, b)$ or $ax + by = a^2 + b^2$.

(40) Find all points which lie on the y-axis and are at a distance 3 from the line whose equation is $3x + 4y = 12$.

$\mathcal{L} = \{ (x, y) : 3x + 4y = 12 \}$

let $a = 3$, $b = 4$, $c = 12$, $x^* = 0$, $y^* = k$

$$\text{dist}(P, \mathcal{L}) = 3 = \frac{|3 \cdot 0 + 4k - 12|}{\sqrt{3^2 + 4^2}} = \frac{|4k - 12|}{5}$$

$|4k - 12| = 15 \longleftrightarrow 4k - 12 = 15$ or $4k - 12 = -15$

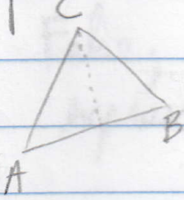
$4k = 27$

$k = \frac{27}{4}$ or $4k = -3$

$k = -\frac{3}{4}$

So the points are $(0, \frac{27}{4})$ and $(0, -\frac{3}{4})$

(41) Given a triangle with vertices A, B, and C; show that $\frac{1}{2}\left|(C-A)\cdot(B-A)_p\right|$ is an expression for the area of the triangle.



Let h_c be the altitude of $\triangle ABC$

then the area is $\frac{1}{2}h_c\|B-A\|$

$$h_c = dist(C, \overline{AB}) = \frac{|(C-A)\cdot(B-A)_p|}{\|(B-A)_p\|}$$

$B-A$ is a direction vector, so $\vec{n} = (B-A)_p \cdot h_c$

$$= dist(C, \overline{AB}) = \frac{|(C-A)\cdot(B-A)_p|}{\|(B-A)_p\|}.$$

But $\|(B-A)_p\| = \|B-A\| = dist(A, B)$

$$\therefore \ Area = \frac{1}{2}\|B-A\|\cdot h_c = \frac{1}{2}\left[\|B-A\| \frac{|(C-A)\cdot(B-A)_p|}{\|(B-A)_p\|}\right]$$

$$= \frac{1}{2}\left[dist(A,B)\cdot\frac{|(C-A)\cdot(B-A)_p|}{dist(A,B)}\right] = \frac{1}{2}\left|(C-A)\cdot(B-A)_p\right|$$

(42) Given $\mathcal{L} = \{(x,y) : ax+by+c=0\}$. Write an equation of a line that is parallel to $\mathcal{L}$ and passes through a point P which is at a distance k from $\mathcal{L}$.

Let $\mathcal{N}$ be the desired line. Then $\mathcal{N} \parallel \mathcal{L}$ implies that (a, b), a normal vector of $\mathcal{L}$, is the normal vector of $\mathcal{N}$.

$$P(x,y) \in \mathcal{N},\ dist(P, \mathcal{L}) = k = \frac{|ax+by+c|}{\sqrt{a^2+b^2}}$$

$$k\sqrt{a^2+b^2} = |ax+by+c|$$

$$ax+by+c = k\sqrt{a^2+b^2}$$

or

$$ax+by+c = -k\sqrt{a^2+b^2}$$

$$\therefore\quad ax+by = -c + k\sqrt{a^2+b^2}$$

$$\text{or}\quad ax+by = -c-k\sqrt{a^2+b^2}$$

$\boxed{5\text{-}9}$ INTERSECTION OF LINES

If $\mathcal{L}_1 = M + q\vec{v}$ and $\mathcal{L}_2 = N + t\vec{w}$ are two lines with a unique intersection point T, which of the following are correct assertions?

1. $(T-M)\cdot\vec{v}_p = 0$ correct
2. $T-M = T-N$ incorrect. $T-M$ is direction vector of $\mathcal{L}_1$, $T-N$ is direction vector of $\mathcal{L}_2$. But $\mathcal{L}_1 \nparallel \mathcal{L}_2$
3. $(T-N)\cdot\vec{w}_p = 0$ correct
4. $T \in (\mathcal{L}_1 \cap \mathcal{L}_2)$ correct
5. $\vec{v}\cdot\vec{w}_p = 0$ incorrect. This would imply $\mathcal{L}_1 \parallel \mathcal{L}_2$
6. $T-M = k\vec{v}$ $(k \neq 0)$ correct

Classify each statement as true or false.
Give supporting reasons.

(7) A determinant is an ordered pair.
False; a determinant is a symbol for an expression which is a real number.

(8) "$x = y$ and $y = 3$" is equivalent to "$(x, y) = (6, 3)$"
This is false because $x = 6 \neq 3 = y$
Hence $(x, y) = (3, 3)$

(9) Given: $\mathcal{L}_1$ and $\mathcal{L}_2$ are two nonparallel lines in the plane, and P and T two distinct points.
If $P \in \{\mathcal{L}_1 \cap \mathcal{L}_2\}$, then
$T \notin \{\mathcal{L}_1 \cap \mathcal{L}_2\}$

True: $P \in \{\mathcal{L}_1 \cap \mathcal{L}_2\}$ implies that P is an intersection point of $\mathcal{L}_1$ and $\mathcal{L}_2$. But nonparallel lines have a unique point of intersection
$\therefore T \notin \{\mathcal{L}_1 \cap \mathcal{L}_2\}$

(10) If $ad \neq bc$, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} c & d \\ a & b \end{vmatrix}$

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ and $\begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad$

False. $ad - bc = bc - ad$ only if $ad = bc$

(11) Evaluate each of the given determinants

TOO EASY! WHY AM I GOING OVER THIS?
REMEMBER that these notes are for posterity!

(11) $\begin{vmatrix} -2 & 3 \\ -6 & 9 \end{vmatrix} = -18-(-18)=0$

(12) $\begin{vmatrix} -2 & 3 \\ 4 & 1 \end{vmatrix} = -2+12 = 10$

(13) $\begin{vmatrix} 5 & 3 \\ -2 & 4 \end{vmatrix} = 20-(-6)=26$

(14) $\begin{vmatrix} 2 & 2 \\ 6 & 6 \end{vmatrix} = 12-12=0$

(15) $$\frac{\begin{vmatrix} 8 & 3 \\ 2 & 1 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ 2 & 3 \end{vmatrix}} = \frac{8-6}{9-8} = 2$$

(16) $$\frac{\begin{vmatrix} 5 & 2 \\ -7 & 2 \end{vmatrix}}{\begin{vmatrix} -1 & 2 \\ -3 & 2 \end{vmatrix}} = \frac{10-(-14)}{-2-(-6)} = \frac{24}{4} = 6$$

(17) $$\frac{\begin{vmatrix} 5 & -3 \\ 6 & 2 \end{vmatrix}}{\begin{vmatrix} 4 & -1 \\ -1 & 2 \end{vmatrix}} = \frac{10-(-18)}{8-1} = \frac{28}{7} = 4$$

(18) $\dfrac{\begin{vmatrix} 4 & 6 \\ 2 & 3 \end{vmatrix}}{\begin{vmatrix} -5 & 3 \\ -1 & 4 \end{vmatrix}} = \dfrac{12-12}{-20-(-3)} = 0$

Determine $\mathcal{L} \cap \mathcal{M}$

(19) $\mathcal{L} = \{(1,4) + t(-1,3)\}$; $\mathcal{M} = \{(4,0) + q(1,2)\}$

$(1,4) + t(-1,3) = (4,0) + q(1,2)$

Taking the inner product of each member of the equation with the vector $(-1,3)_p = (-3,-1)$, we find

$(1,4)\cdot(-3,-1) + t(-1,3)\cdot(-3,-1) = (4,0)\cdot(-3,-1) + q(1,2)\cdot(-3,-1)$

$-7 + t(0) = -12 - 5q \leftrightarrow 5 = -5q \leftrightarrow q = -1$

Thus $\mathcal{L} \cap \mathcal{M} = \{(4,0) + (-1)(1,2)\} = \{(4,0) + (-1,-2)\}$
$= \{(3,-2)\}$

(20) $\mathcal{L} = \{(8,-3) + t(-5,2)\}$; $\mathcal{M} = \{(4,10) + q(2,3)\}$

$(8,-3) + t(-5,2) = (4,10) + q(2,3)$

Multiplying by $\vec{n}_{\mathcal{L}} = (-5,2)_p = (-2,-5)$:

$(8,-3)\cdot(-2,-5) + t(-5,2)\cdot(-2,-5) = (4,10)\cdot(-2,-5) + q(2,3)\cdot(-2,-5)$

$-1 + 0 = -58 - 19q \leftrightarrow 57 = -19q \leftrightarrow q = -3$

$x = 4 + (-3)(2) = -2$

$y = 10 + (-3)(3) = 1$

$\therefore \mathcal{L} \cap \mathcal{M} = \{-2, 1\}$

(21) $\mathcal{L} = \{(3-2t, 1-t)\}$; $\mathcal{M} = \{(4+5q, 2+3q)\}$

$\mathcal{L} = \{(3,1) + t(-2,-1)\}$

$\mathcal{M} = \{(4,2) + q(5,3)\}$

Set $\mathcal{L} = \mathcal{M}$

$(3,1) + t(-2,-1) = (4,2) + q(5,3)$

Multiplying by $\vec{n}_{\mathcal{L}} = (-2,-1)_p = (1,-2)$, we find

$(3,1)\cdot(1,-2) + t(-2,-1)\cdot(1,-2) = (4,2)\cdot(1,-2) + q(5,3)\cdot(1,-2)$

$1 + 0 = 0 - q \longleftrightarrow q = -1$

$x = 4 - 5 = -1$

$y = 2 - 3 = -1$

$\therefore \mathcal{L} \cap \mathcal{M} = \{(-1,-1)\}$

To check that $(-1,-1) \in \mathcal{L}$ and $(-1,-1) \in \mathcal{M}$,
verify that $[(-1,-1) - (3,1)]\cdot(-2,-1)_p = 0$
and $[(-1,-1) - (4,2)]\cdot(5,3)_p = 0$

$(-4,-2)\cdot(1,-2) = -4+4 = 0$ ✓

$(-5,-3)\cdot(-3,5) = 15-15 = 0$ ✓

(22) $\mathcal{L} = \{(3+3t, \frac{2}{3}-2t)\}$; $\eta = \{(4+2q, 5+3q)\}$

$\mathcal{L} = \{(3, \frac{2}{3}) + t(3,-2)\}$

$\eta = \{(4,5) + q(2,3)\}$

$(3, \frac{2}{3}) + t(3,-2) = (4,5) + q(2,3)$

"Multiplying by" $\vec{n}_{\mathcal{L}} = (3,-2)_p = (2,3)$:

~~means~~ "taking the inner product of each member of this equation with the vector $\vec{n}_{\mathcal{L}}$"

$(3, \frac{2}{3})\cdot(2,3) + t(3,-2)\cdot(2,3) = (4,5)\cdot(2,3) + q(2,3)\cdot(2,3)$

$8 + t(0) = 23 + 13q \longleftrightarrow -15 = 13q \longleftrightarrow q = -\frac{15}{13}$

so $x = 4 + (-\frac{15}{13})(2) = 4 - \frac{30}{13} = \frac{52-30}{13} = \frac{22}{13}$

and $y = 5 + (-\frac{15}{13})(3) = 5 - \frac{45}{13} = \frac{65-45}{13} = \frac{20}{13}$

$\therefore \mathcal{L} \cap \eta = \{(\frac{22}{13}, \frac{20}{13})\}$

To check that $(\frac{22}{13}, \frac{20}{13}) \in \mathcal{L}$ and $(\frac{22}{13}, \frac{20}{13}) \in \eta$

verify that $[(\frac{22}{13}, \frac{20}{13}) - (3, \frac{2}{3})]\cdot(3,-2)_p = 0$

and $[(\frac{22}{13}, \frac{20}{13}) - (4,5)]\cdot(2,3)_p = 0$

$(\frac{22}{13} - 3, \frac{20}{13} - \frac{2}{3})\cdot(2,3) = (-\frac{17}{13}, \frac{34}{39})\cdot(2,3) = \frac{-34}{13} + \frac{34}{13} = 0$ ✓

$(-\frac{30}{13}, \frac{-45}{13})\cdot(-3,2) = \frac{90}{13} - \frac{90}{13} = 0$ ✓

(23) $\mathcal{L} = \{(6+3q,\ 3-2q)\}$.
$\eta = \{(3,5) + s(-3,2)\}$

$\mathcal{L} = \{(6,3) + q(3,-2)\}$; $\vec{n}_{\mathcal{L}} = (3,-2)_p = (2,3)$

$(6,3) + q(3,-2) = (3,5) + s(-3,2)$

Taking the inner product of each member of this equation with the vector $\vec{n}_{\mathcal{L}}$, we find

$(6,3)\cdot(2,3) + q(3,-2)\cdot(2,3) = (3,5)\cdot(2,3) + s(-3,2)\cdot(2,3)$

$21 + 0 = 21 + 0 \leftrightarrow 21 = 21$ (These lines coincide!)

What does this mean? Well, notice that both $\mathcal{L}$ and η have the same direction vector $(3,-2)$.

$\vec{v}_{\eta} = s(-3,2) = -(-3,2) = (3,-2)$

$\therefore\ \mathcal{L} \parallel \eta$. When $q = 0$, $(6,3) \in \mathcal{L}$

$(6,3) \in \eta$ when $6 = 3 - 3s \leftrightarrow s = -1$

Since the lines are parallel and have a common point, they coincide. $\mathcal{L} \cap \eta = \mathcal{L}$.

(24) $\mathcal{L} = \{(2,-1) + q(1,-4)\}$; $\mathcal{M} = \{s(1,-4)\}$

$\vec{n}_{\mathcal{L}} = (1,-4)_p = (4,1)$

$(2,-1)\cdot(4,1) + q(1,-4)\cdot(4,1) = 0 + s(1,-4)\cdot(4,1)$

$9 + 0 = 0 - 0 \leftrightarrow 9 = 0$

What does this mean? There is no intersection point. $\mathcal{L} \parallel \mathcal{M}$ and $\mathcal{L} \cap \mathcal{M} = \phi$

Both $\mathcal{L}$ and $\mathcal{M}$ have $(1,-4)$ as a direction vector, and $\mathcal{L} \parallel \mathcal{M}$. $(2,-1) \in \mathcal{L}$ when $q = 0$.

$(2,-1) = s(1,-4)$ when $2 = s$ and $-1 = -4s$ which is impossible $\therefore$ $\mathcal{L}$ and $\mathcal{M}$ cannot coincide $\therefore$ $\mathcal{L} \cap \mathcal{M} = \phi$.

Solve the given system of simultaneous linear equations by (a) by taking the inner product of each member of the equation with vector normal to one of the lines

(b) using determinants

(25) $\boxed{A}$

$x + 4y = 27$ $\qquad (x+4y, x+2y) = (27, 21)$

$x + 2y = 21$ $\qquad (1,1)x + (4,2)y = (27,21)$

Taking the inner product of this equation first with $(4,2)_p = (-2,4)$, and then with $(1,1)_p = (-1,1)$,

(1) $(-2,4)\cdot(1,1)x + (-2,4)\cdot(4,2)y = (-2,4)\cdot(27,21)$

$2x + 0 = 30 \longleftrightarrow x = 15$

$(1,1)x + (4,2)y = (27,21)$

(2): $(-1,1)\cdot(1,1)x + (-1,1)\cdot(4,2)y = (-1,1)\cdot(27,21)$

$$0 - 2y = -6 \leftrightarrow y = 3$$

Thus, the only possible solution of the given system is $(-15, 3)$.

To check: $x + 4y = 27$: $15 + 4(3) = 27$ ✓

$x + 2y = 21$: $15 + 2(3) = 21$ ✓

$\therefore (x,y) = (15, 3)$

$\boxed{B}$ (using determinants)

$$D = \begin{vmatrix} 1 & 4 \\ 1 & 2 \end{vmatrix} = 2 - 4 = -2 \neq 0$$

$$\therefore \quad x = -\frac{1}{2}\begin{vmatrix} 27 & 4 \\ 21 & 2 \end{vmatrix} = -\frac{1}{2}(54 - 84) = 15$$

$$y = -\frac{1}{2}\begin{vmatrix} 1 & 27 \\ 1 & 21 \end{vmatrix} = -\frac{1}{2}(21 - 27) = 3$$

$(x,y) = (15,3)$

(26) $\begin{aligned} 2x - 3y &= 1 \\ 3x - 4y &= 7 \end{aligned}$

Method A: $(2x-3y,\ 3x-4y) = (1,7)$

$(2,3)x + (-3,-4)y = (1,7)$

(1) take inner product of each member with vector $(-3,-4)_p$
$= (4,-3)$:

$(4,-3)\cdot(2,3)x + (4,-3)\cdot(-3,-4)y = (4,-3)\cdot(1,7)$

$-x + 0 = -17 \leftrightarrow x = 17$

(2) take inner product of each member with vector $(2,3)_p$
$= (-3,2)$:

$(-3,2)\cdot(2,3)x + (-3,2)\cdot(-3,-4)y = (-3,2)\cdot(1,7)$

$0 + y = 11 \leftrightarrow y = 11$

$\therefore (x,y) = (17,11)$

Method B: $D = \begin{vmatrix} 2 & -3 \\ 3 & -4 \end{vmatrix} = -8-(-9) = 1$

$x = 1 \begin{vmatrix} 1 & -3 \\ 7 & -4 \end{vmatrix} = -4-(-21) = 17$

$y = 1 \begin{vmatrix} 2 & 1 \\ 3 & 7 \end{vmatrix} = 14-3 = 11$

$\therefore (x,y) = (17,11)$

27 $\quad 2x + 5y = 18$
$\quad\quad 3x + 4y = 27$

Method A: $(2x+5y,\ 3x+4y) = (18, 27)$

$$(2,3)x + (5,4)y = (18, 27)$$

(1) Take inner product of each member of the equation with normal vector $(5,4)_p = (-4,5)$:

$$(-4,5)\cdot(2,3)x + (-4,5)\cdot(5,4)y = (-4,5)\cdot(18,27)$$

$$7x + 0 = -72 + 135 = 63 \leftrightarrow x = 9$$

(2) Take inner product of each member of the equation with normal vector $(2,3)_p = (-3,2)$:

$$(-3,2)\cdot(2,3)x + (-3,2)\cdot(5,4)y = (-3,2)\cdot(18,27)$$

$$0 - 7y = -54 + 54 = 0 \leftrightarrow y = \frac{0}{-7} = 0$$

$$\therefore (x,y) = (9,0)$$

Method B: $D = \begin{vmatrix} 2 & 5 \\ 3 & 4 \end{vmatrix} = 8 - 15 = -7$

$$x = -\frac{1}{7}\begin{vmatrix} 18 & 5 \\ 27 & 4 \end{vmatrix} = -\frac{1}{7}(72 - 135) = \frac{63}{7} = 9$$

$$y = -\frac{1}{7}\begin{vmatrix} 2 & 18 \\ 3 & 27 \end{vmatrix} = -\frac{1}{7}(54 - 54) = 0$$

$$\therefore (x,y) = (9,0)$$

(28) $\frac{x}{2} + \frac{y}{3} = 2 \qquad \frac{x}{3} + \frac{y}{9} = 1$

First I rewrite these equations to remove the fractions.
Multiplying the first equation by 6: $3x + 2y = 12$

Multiplying the second equation by 9: $3x + y = 9$

Method A: $(3x + 2y,\ 3x + y) = (12, 9)$

$(3,3)x + (2,1)y = (12, 9)$

(1) Taking inner product of each member of the equation with normal vector $(2,1)_p = (-1,2)$:

$(-1,2)\cdot(3,3)x + (-1,2)\cdot(2,1)y = (-1,2)\cdot(12,9)$

$3x + 0 = -12 + 18 = 6 \leftrightarrow x = 2$

(2) Taking inner product of each member of the equation with normal vector $(3,3)_p = (-3,3)$:

$(-3,3)\cdot(3,3)x + (-3,3)\cdot(2,1)y = (-3,3)\cdot(12,9)$

$0 - 3y = -36 + 27 = -9 \leftrightarrow y = 3$

$\therefore (x,y) = (2,3) \quad \left[\frac{2}{2} + \frac{3}{3} = 2 \checkmark \quad \frac{2}{3} + \frac{3}{9} = 1 \checkmark\right]$

Method B: $D = \begin{vmatrix} 3 & 2 \\ 3 & 1 \end{vmatrix} = 3 - 6 = -3$

$x = -\frac{1}{3}\begin{vmatrix} 12 & 2 \\ 9 & 1 \end{vmatrix} = -\frac{1}{3}(12 - 18) = -\frac{1}{3}(-6) = 2$

$y = -\frac{1}{3}\begin{vmatrix} 3 & 12 \\ 3 & 9 \end{vmatrix} = -\frac{1}{3}(27 - 36) = -\frac{1}{3}(-9) = 3$

$\therefore (x,y) = (2,3)$

(29) $2x + 3y = 19$
$x - y = 12$

Method A: $(2x+3y,\ x-y) = (19, 12)$
$(2,1)x + (3,-1)y = (19,12)$

Note: this notation is very formal. One might also write $\begin{bmatrix} 2x+3y \\ x-y \end{bmatrix} = \begin{bmatrix} 19 \\ 12 \end{bmatrix}$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} x + \begin{bmatrix} 3 \\ -1 \end{bmatrix} y = \begin{bmatrix} 19 \\ 12 \end{bmatrix}$$

Look more "familiar"?

(1) Taking inner product of each member of the equation with normal vector $(3,-1)_p = (1,3)$:

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \end{bmatrix} y = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 19 \\ 12 \end{bmatrix}$$

$$5x + 0 = 19 + 36 = 55 \leftrightarrow x = 11$$

(2) Taking inner product of each member of the equation with normal vector $(2,1)_p = (-1,2) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} x + \begin{bmatrix} -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \end{bmatrix} y = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 19 \\ 12 \end{bmatrix}$$

$$0 - 5y = -19 + 24 = 5 \longleftrightarrow y = -1$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ -1 \end{bmatrix}$$

Method B: $D = \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} = -2-3 = -5$

$$x = -\frac{1}{5}\begin{vmatrix} 19 & 3 \\ 12 & -1 \end{vmatrix} = -\frac{1}{5}(-19-36) = -\frac{1}{5}(-55) = 11$$

$$y = -\frac{1}{5}\begin{vmatrix} 2 & 19 \\ 1 & 12 \end{vmatrix} = -\frac{1}{5}(24-19) = -\frac{1}{5}(5) = -1$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ -1 \end{bmatrix}$$

Note; since this text uses notation $(x, y) = (11, -1)$, I will use this notation. I just wanted to show this in that more familiar form.

Still, I admit that "method A" is quite exotic to me, even at age 50.

The common method would be to just use elimination and substitution:

$$\begin{array}{r} 2x + 3y = 19 \\ -2x + 2y = -24 \\ \hline 0 + 5y = -5 \end{array}$$

(I multiply second equation by -2)

$y = -1$ and substitute $2x + 3(-1) = 19$

$2x = 22$

$x = 11$ $\therefore (x, y) = (-1, 11)$

(30) $3x - 4y = 0$ 　　Am I a robot?
$2x + y = 33$ 　　I am an ape.

Common method first! (says the ape man)

Let's have a little fun and try solving this with elimination and substitution.

ELIMINATION

$3x - 4y = 0$

$\underline{8x + 4y = 132}$

$11x = 132$

$x = \frac{132}{11} = 12$

SUBSTITUTION

$3(12) - 4y = 0$

$36 = 4y$

$y = 9$

so $(x, y) = (12, 9)$

check: $2(12) + 9 = 24 + 9 = 33$ ✓

(Now, continue in robot mode)

Method A:

$(3x - 4y, 2x + y) = (0, 33)$

$(3,2)x + (-4,1)y = (0,33)$

(1) Taking the inner product of each member of the equation with normal vector $(-4,1)_p = (-1,-4)$

$(-1,-4)\cdot(3,2)x + (-1,-4)\cdot(-4,1)y = (-1,-4)\cdot(0,33)$

$-11x + 0 = -132 \longleftrightarrow x = \frac{-132}{-11} = 12$

(2) Taking the inner product of each member of the equation with normal vector $(3,2)_p = (-2,3)$:

$(-2,3)\cdot(3,2)x + (-2,3)\cdot(-4,1)y = (-2,3)\cdot(0,33)$

$0 + 11y = 99 \leftrightarrow y = 9 \therefore (x,y) = (12,9)$

Method B (using determanants in a robotic manner)

$$D = \begin{vmatrix} 3 & -4 \\ 2 & 1 \end{vmatrix} = 3-(-8) = 3+8 = 11$$

$$x = \frac{1}{11}\begin{vmatrix} 0 & -4 \\ 33 & 1 \end{vmatrix} = \frac{1}{11}(0-(-132)) = \frac{132}{11} = 12$$

$$y = \frac{1}{11}\begin{vmatrix} 3 & 0 \\ 2 & 33 \end{vmatrix} = \frac{1}{11}(99-0) = 9$$

$$\therefore (x,y) = (12,9)$$

Note: Before this scratchpad is full, I will point out for one last time that this old school notation means the same as $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \\ 9 \end{bmatrix}$.

ordered pair (x,y) is vector $\begin{bmatrix} x \\ y \end{bmatrix}$.

(31) $8x + 5y = 18$
$3x + 4y = 27$

Method A: $(8x+5y, 3x+4y) = (18, 27)$

$(8,3)x + (5,4)y = (18,27)$

(1) $(5,4)_p = (-4,5)$

$(-4,5)\cdot(8,3)x + (-4,5)\cdot(5,4)y = (-4,5)\cdot(18,27)$

$(-32+15)x + 0 = -72 + 135$

$-17x = 63 \leftrightarrow x = -\frac{63}{17}$

(2) $(8,3)_p = (-3,8)$

$(-3,8)\cdot(8,3)x + (-3,8)\cdot(5,4)y = (-3,8)\cdot(18,27)$

$0 + 17y = 162 \leftrightarrow y = 162/17$

$(x, y) = \left(-\frac{63}{17}, \frac{162}{17}\right)$

Method B: $D = \begin{vmatrix} 8 & 5 \\ 3 & 4 \end{vmatrix} = 32 - 15 = 17$

$x = \frac{1}{17}\begin{vmatrix} 18 & 5 \\ 27 & 4 \end{vmatrix} = \frac{1}{17}(72 - 135) = \frac{1}{17}(-63) = \frac{-63}{17}$

$y = \frac{1}{17}\begin{vmatrix} 8 & 18 \\ 3 & 27 \end{vmatrix} = \frac{1}{17}(216 - 54) = \frac{1}{17}(162) = \frac{162}{17}$

$\therefore \quad (x, y) = \left(-\frac{63}{17}, \frac{162}{17}\right)$

Observation: Although more "mechanical", method B is arithmetically less difficult.

(32) $x + 3y = 26$ [NEW DAY]

$2x - 6y = 52$

Rather than racing through robotically, it might elucidate the differences in what I have named methods A and B to first solve this the most "common" way. That is, by either multiplying the first equation by -2 or by multiplying the first equation by +2, to eliminate either the unknown variable x or y, respectively then solving by substitution.

$$\begin{aligned} -2x - 6y &= -52 \\ 2x - 6y &= 52 \\ \hline 0 \quad -12y &= 0 \leftrightarrow y = 0 \end{aligned}$$

$x = 26$ so $(x, y) = (26, 0)$

$$(x+3y,\ 2x-6y) = (26,\ 52)$$

$$12\overline{)312}\ \ 26$$
$$\frac{24}{72},\ \frac{72}{0}$$

Method A: $(1,2)x + (3,-6)y = (26,52)$

(1) $(3,-6)_p = (6,3)$

$$(6,3)\cdot(1,2)x + (6,3)\cdot(3,-6)y = (6,3)\cdot(26,52)$$

$$12x + 0 = 156 + 156 = 312 \leftrightarrow x = \frac{312}{12} = 26$$

(2) $(1,2)_p = (-2,1)$

$$(-2,1)\cdot(1,2)x + (-2,1)\cdot(3,-6)y = (-2,1)\cdot(26,52)$$

$$0 - 12y = -52 + 52 = 0 \leftrightarrow y = \frac{0}{-12} = 0$$

$$\therefore (x,y) = (26,0)$$

Method B: $D = \begin{vmatrix} 1 & 3 \\ 2 & -6 \end{vmatrix} = -6-6 = -12$

$$x = -\frac{1}{12}\begin{vmatrix} 26 & 3 \\ 52 & -6 \end{vmatrix} = -\frac{1}{12}(-156-156) = -\frac{1}{12}(-312) = 26$$

$$y = -\frac{1}{12}\begin{vmatrix} 1 & 26 \\ 2 & 52 \end{vmatrix} = -\frac{1}{12}(52-52) = 0$$

$$\therefore (x,y) = (26,0)$$

(33) Find the vertices of a triangle whose sides are segments of the three lines specified by $3x + 2y - 2 = 0$; $x - 3y - 6 = 0$; $x + 3y + 12 = 0$

How to approach this problem? Find where these lines intersect and you will have discovered the vertices. ~ ape-man

RIGHT YOU ARE!

$\mathcal{L} = \{(x,y): 3x+2y = 2\}$

$\mathcal{M} = \{(x,y): x-3y = 6\}$

$\mathcal{N} = \{(x,y): x+3y = -12\}$

① Find $\mathcal{L} \cap \mathcal{M}$

$(3x+2y,\ x-3y) = (2,6);\quad (3,1)x + (2,-3)y = (2,6)$

$(2,-3)_p = (3,2)$

$(3,2)\cdot(3,1)x + (3,2)\cdot(2,-3)y = (3,2)\cdot(2,6)$

$11x + 0 = 6+12 = 18 \leftrightarrow x = \frac{18}{11}$

$(3,1)_p = (-1,3)$

$(-1,3)\cdot(3,1)x + (-1,3)\cdot(2,-3)y = (-1,3)\cdot(2,6)$

$0 - 11y = 16 \leftrightarrow y = -\frac{16}{11}$

$\therefore \mathcal{L} \cap \mathcal{M} = \{(18/11,\ -16/11)\}$

② Find $\mathcal{L} \cap \mathcal{N}$

$(3x+2y,\ x+3y) = (2,-12);\quad (3,1)x + (2,3)y = (2,-12)$

$(2,3)_p = (-3,2)$

$(-3,2)\cdot(3,1)x + (-3,2)\cdot(2,3)y = (-3,2)\cdot(2,-12)$

$-7x + 0 = -6-24 = -30 \leftrightarrow x = 30/7$

$(3,1)_p = (-1,3)$

$(-1,3)\cdot(3,1)x + (-1,3)\cdot(2,3)y = (-1,3)\cdot(2,-12)$

$0 + 7y = -2-36 = -38 \leftrightarrow y = -\frac{38}{7}$

$\therefore \mathcal{L} \cap \mathcal{N} = \{(\frac{30}{7}, -\frac{38}{7})\}$

(3) Find $\mathcal{M} \cap \mathcal{N}$

$(x-3y,\ x+3y) = (6,-12)$; $(1,1)x + (-3,3)y = (6,-12)$

$(-3,3)_p = (-3,-3)$

$(-3,-3)\cdot(1,1)x + (-3,-3)\cdot(-3,3)y = (-3,-3)\cdot(6,-12)$

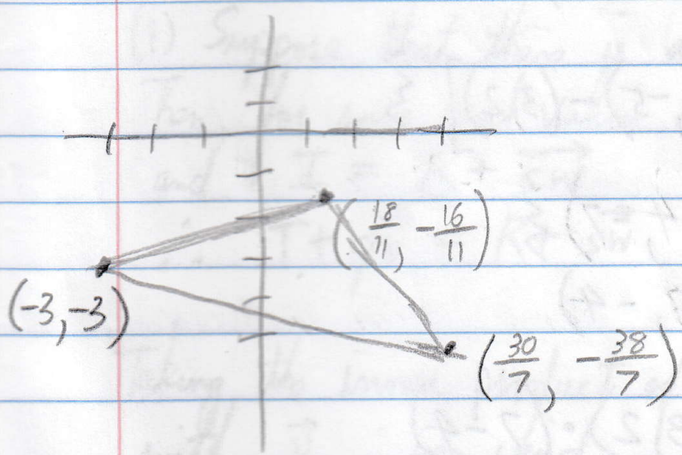
$-6x + 0 = -18 + 36 = 18 \longleftrightarrow x = -\frac{18}{6} = -3$

$(1,1)_p = (-1,1)$

$(-1,1)\cdot(1,1)x + (-1,1)\cdot(-3,3)y = (-1,1)\cdot(6,-12)$

$0 + 6y = -6-12 = -18 \longleftrightarrow y = \frac{-18}{6} = -3$

$\therefore \mathcal{M} \cap \mathcal{N} = \{(-3,-3)\}$



Of course, you could have used determinants or elimination and substitution. I prefer using normal vectors and inner products since it is a rather obscure technique.

(validates documentation for posterity).

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(34) Find the equation of the line which passes through $(3,2)$ and the intersection of the lines $2x - y = 3$ and $3x - 2y = 7$.

Strategy: Find intersection and let it be I.

Let $T = (3,2)$. Then $\mathcal{L} = \{T + t(I - T)\}$. We'll see...

$2x - y = 3$
$3x - 2 = 7$

$$D = \begin{vmatrix} 2 & -1 \\ 3 & -2 \end{vmatrix} = -4 - (-3) = -1$$

Use determinants to solve.

$$x = -\begin{vmatrix} 3 & -1 \\ 7 & -2 \end{vmatrix} = -(-6 - (-7)) = -(1) = -1$$

$$y = -\begin{vmatrix} 2 & 3 \\ 3 & 7 \end{vmatrix} = -(14 - 9) = -5$$

So, intersection point $I = (-1, -5)$

$$\mathcal{L} = \{ (3,2) + t[(-1,-5) - (3,2)] \}$$

$$= \{ (3,2) + t(-4,-7) \}$$

$$\vec{n} = (-4,-7)_p = (7,-4)$$

$$(x,y) \cdot (7,-4) = (3,2) \cdot (7,-4)$$

(because $X \cdot \vec{n} = T \cdot \vec{n}$)

$$\boxed{7x - 4y = 13}$$

(35) Prove: Nonparallel lines in a plane have a unique point of intersection.

Let $\mathcal{L} = \{T + q\vec{V}\}$ and $\mathcal{M} = \{R + s\vec{W}\}$ be nonparallel lines. We must show that

(1) there is only one point that can possibly be on both $\mathcal{L}$ and $\mathcal{M}$;

(2) This point actually does belong to both lines.

Since $\vec{V}$ and $\vec{W}$ are not parallel, $\vec{W} \cdot \vec{V}_p \neq 0$

(1) Suppose that there is a point I common to $\mathcal{L}$ and $\mathcal{M}$. Then, for some real values of q and s, $I = T + q\vec{V}$ and $I = R + s\vec{W}$

$\therefore \quad T + q\vec{V} = R + s\vec{W}$.

Taking the inner product of each member of this equation with $\vec{V}_p$, you find $T \cdot \vec{V}_p + q\vec{V} \cdot \vec{V}_p = R \cdot \vec{V}_p + s\vec{W} \cdot \vec{V}_p$

$$T \cdot \vec{V}_p + 0 = R \cdot \vec{V}_p + s(\vec{W} \cdot \vec{V}_p)$$

because $\vec{W} \cdot \vec{V}_p \neq 0$, you may rewrite this equation in the form

$$\frac{(T-R) \cdot \vec{V}_p}{\vec{W} \cdot \vec{V}_p} = s$$

Thus, $I = R + \left[\dfrac{(T-R) \cdot \vec{V}_p}{\vec{W} \cdot \vec{V}_p}\right]\vec{W}$ is the only point that can lie on both $\mathcal{L}$ and $\mathcal{M}$.

(2) The point $I = R + \left[\dfrac{(T-R)\cdot\vec{V}_p}{\vec{W}\cdot\vec{V}_p}\right]\vec{W}$ certainly belongs to $\mathcal{M}$. Verifying that it belongs to $\mathcal{L}$ is equivalent to checking that

$(I-T)\cdot\vec{V}_p = 0.$

But, $(I-T)\cdot\vec{V}_p = (R-T)\cdot\vec{V}_p + \left[\dfrac{(T-R)\cdot\vec{V}_p}{\vec{W}\cdot\vec{V}_p}\right]\vec{W}\cdot\vec{V}_p$

$= (R-T)\cdot\vec{V}_p + (T-R)\cdot\vec{V}_p$

$= (R-T)\cdot\vec{V}_p - (R-T)\cdot\vec{V}_p = 0$

$\therefore\ (I-T)\cdot\vec{V}_p = 0$; Hence $I \in \mathcal{L}$ and $I \in \mathcal{M}$

$\therefore$ nonparallel lines in the plane have a unique point of intersection.

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36 Given the system $\begin{cases} a_1 x + b_1 y = c_1 \\ a_2 x + b_2 y = c_2 \end{cases}$

where $a_1 b_2 - a_2 b_1 \neq 0$.

Prove that $x = \dfrac{c_1 b_2 - c_2 b_1}{a_1 b_2 - a_2 b_1}$, $y = \dfrac{a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$

$$(a_1x + b_1y,\ a_2x + b_2y) = (c_1, c_2)$$

$$(a_1, a_2)x + (b_1, b_2)y = (c_1, c_2)$$

Taking inner product of each member of this equation, first with $(b_1, b_2)_p = (-b_2, b_1)$, and then with $(a_1, a_2)_p = (-a_2, a_1)$, you find

$$(-b_2, b_1)\cdot(a_1, a_2)x + (-b_2, b_1)\cdot(b_1, b_2)y = (-b_2, b_1)\cdot(c_1, c_2)$$

$$(-a_1b_2 + a_2b_1)x + 0 = -c_1b_2 + c_2b_1$$

$$x = \frac{-c_1b_2 + c_2b_1}{-a_1b_2 + a_2b_1} = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}$$

and

$$(-a_2, a_1)\cdot(a_1, a_2)x + (-a_2, a_1)\cdot(b_1, b_2)y = (-a_2, a_1)\cdot(c_1, c_2)$$

$$0 + (-a_2b_1 + a_1b_2)y = -a_2c_1 + a_1c_2$$

$$y = \frac{-a_2c_1 + a_1c_2}{-a_2b_1 + a_1b_2} = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

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(37) Given: $P(x_0, y_0)$ as the intersection point of $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$,

Show that P lies on the line specified by

$$k(a_1x + b_1y + c_1) + t(a_2x + b_2y + c_2) = 0$$

where k and t are any two real numbers, not both 0.

$P(x_0, y_0)$ is an intersection point of $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ implies

$$a_1x + b_1y + c_1 = a_2x + b_2y + c_2 = 0.$$

P lies on the line specified by

$$k(a_1x + b_1y + c_1) + t(a_2x + b_2y + c_2) = 0$$

if and only if $k(a_1x_0 + b_1y_0 + c_1)$

$$+ t(a_2x_0 + b_2y_0 + c_2) = 0;$$

but $k(a_1x_0 + b_1y_0 + c_1) = k \cdot 0,$

$$t(a_2x_0 + b_2y_0 + c_2) = t \cdot 0$$

$\therefore k(a_1x_0 + b_1y_0 + c_1) + t(a_2x_0 + b_2y_0 + c_2)$

$$= 0 + 0 = 0$$

$\therefore P$ lies on the line specified by

$$k(a_1x + b_1y + c_1) + t(a_2x + b_2y + c_2) = 0.$$

If k and t both equal zero,

$$k(a_1x + b_1y + c_1) + t(a_2x + b_2y + c_2) = 0$$

does not specify a line.

EXERCISES 38, 39, 40 of 5-9 will continue in SCRATCHPAD #4 OF THIS SERIES